Phase Separation on Biological Membranes

DISSERTATION ZUR ERLANGUNG DES DOKTORGRADES
DER NATURWISSENSCHAFTEN (DR. RER. NAT.)
AN DER FAKULTÄT FÜR MATHEMATIK
DER UNIVERSITÄT REGENSBURG

vorgelegt von

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im Jahr 2018

Promotionsgesuch eingereicht am: 23. März 2018

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Abstract

We provide a detailed mathematical analysis of a model for lipid raft formation in cell membranes which was recently proposed by Garcke, Rätz, Röger and the author. Lipid rafts are domains of a specific molecule composition (mostly saturated lipids and cholesterols) in biological membranes. In principle, the proposed model is a phase-field model describing phase separation between saturated and unsaturated lipids. Additionally, the model is based the assumption that active transport processes of cholesterols into and out of the membrane influence the phase separation within the membrane, due to a high affinity between cholesterols and saturated lipids. As such, the model takes the form of an extended Cahn-Hilliard equation which contains additional terms to account for the cholesterol transport.

We prove results on the existence and regularity of solutions, their long-time behaviour, and on the existence of stationary solutions. Moreover, we investigate three different asymptotic regimes. The first two are connected to model parameters: We study the case of large cytosolic diffusion and investigate the effect of a infinitely large affinity between cholesterols and saturated lipids. The third is a detailed analysis of the sharp-interface limit of the phase-field model.

The first case leads to the reduction of coupled bulk-surface equations in the lipid raft model to a system of surface equations with non-local contributions. Subsequently, we recover the well-known Ohta-Kawasaki equations as the limit for infinitely large affinity between cholesterols and saturated lipids.

We prove the convergence of solutions of the lipid raft model to weak solutions of the sharp-interface limit in the sense of varifolds. Finally, we directly prove the existence of weak solutions to the sharp-interface limit.

Zusammenfassung

Diese Arbeit bietet eine detaillierte mathematische Anaylse für ein Modell zur Bildung von so genannten Lipid Rafts (dt.: Lipidflöße), das kürzlich von Garcke, Rätz, Röger und dem Autor vorgeschlagen wurde. Bei Lipid Rafts handelt es sich um Gebiete mit einer spezifischen molekularen Zusammensetzung (größtenteils Cholesterol und gesättigte Lipide) in Zellmembranen. Grundsätzlich handelt es sich um ein Phasenfeldmodell zur Beschreibung der Phasentrennung

zwischen gesättigten und ungesättigten Lipiden. Zusätzlich beruht das Modell auf der Annahme, dass wegen der starken Bindung zwischen Cholesterol und gesättigten Lipiden ein aktiver Cholesteroltransport in und aus der Membran die Phasentrennung innerhalb der Membran beinflusst. Dementsprechend handelt es sich bei dem Modell um ein erweitertes Cahn-Hilliard System, das um Terme ergänzt wird die den Cholesteroltransport modellieren.

Wir beweisen Resultate zur Existenz und Regularität von Lösungen, zum Langzeitverhalten und zur Existenz stationärer Lösungen. Außerdem untersuchen wir das asymptotische Verhalten in drei unterscheiedlichen Fällen. Die ersten beiden sind mit Modellierungsparametern verknüpft: Wir studieren große Diffusionskonstanten im Cytosol und untersuchen die Auswirkungen von infinitesimal großer Bindung zwischen Cholesterol und gesättigten Lipiden. Der dritte Fall betrifft den scharfen Interface Limes des Phasenfeldmodells.

Im ersten Fall erhalten wir ein reduziertes Modell, in dem die gekoppelten Gleichungen auf der Oberfläche und im Inneren durch Oberflächengleichungen mit nicht lokalen Anteilen ersetzt wurden. Anschließend erhalten wir im Grenzwert für unendlich große Bindung zwischen Cholesterol und gesättigten Lipiden die bekannten Ohta-Kawasaki Gleichungen.

We beweisen die Konvergenz von Lösungen des Lipid Raft Modells gegen schwachen Lösungen des scharfen Interface Limes. Die Grenzfläche wird dabei als Varigfaltigkeit formuliert. Schließlich zeigen wir die Exstenz von Lösungen des scharfen Interface Limes mit Hilfe einer direkten Methode.

Acknowledgements

I would like to express my heartfelt gratitude to my supervisor, Prof. Dr. Helmut Abels, who gave me the opportunity to work on this thesis. He invested a lot of time in helpful discussions (always on short notice!) and his huge experience in the field of partial differential equations was of tremendous help to me. I am grateful for his outstanding support and all of his efforts he invested in this thesis.

Prof. Dr. Harald Garcke took a lot of interest in my work and his important contributions to the joint reaserach article [GKRR16], which was the first step in this research, are greatfully acknowleged. I profitted a lot from his insights into mathematical modelling.

I am equally grateful to Prof. Dr. Matthias Röger and PD Dr. Andreas Rätz for their part in writing the aforementioned article. Central ideas in [GKRR16] have their roots in joint discussions during a visit to Dortmund. Both were also important teachers during my studies at the TU Dortmund and Prof. Dr. Matthias Röger suggested the fascinating question of phase separation on biological membranes to me, which proved to be connected to many interesting mathematical topics.

My list of important teachers would not be complete without Dr. Danielle Hilhorst, who shared her deep insight in phase field models and singular limits and was an invaluable guide to the academic world.

I would like to extend my thanks to everyone in the applied analysis working group in Regensburg. Our collobarative atmosphere was always rewarding and encouraging. I am particularly grateful to Kei-Fong Lam, PhD, for many discussions on (modified) Cahn-Hilliard equations and sharp-interface limits and (in alphabetical order) to Julia Butz, Michael Gößwein, and Dr. Christine Pfeuffer, who always found time for discussions and to answer my questions.

I am fortunate to have a loving and supportive family that made it always look natural to be curious and to ask about science, art and music. It is impossible to overestimate how much I value everything they have done for me, including (but of course not limited to) the tedious task of proof reading this thesis.

This work was partially supported by the SPP 1506 ”Transport Processes at Fluidic Interfaces” of the German Science Foundation (DFG) through the grant AB285/4-2. This support is gratefully acknowledged.

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Chapter 1

Introduction

While the most common example of phase separation might be an emulsion of water and oil separating over time, phase separation can be observed in various physical or biological systems. Examples include not only emulsions but also metallic alloys and synthetic materials like diblock copolymers.

For our purpose, a phase is considered to be a domain in space in which physical parameters like density and the chemical composition of a material are homogeneous. We speak of phase separation if an inhomogeneous mixture of several components separates over time into a configuration that displays different phases.

If the initial state of the considered system is a mixture of different components which are subject to phase separation, the evolution first rapidly decomposes the mixture into several distinct domains which are homogeneous in one component. Subsequently, the evolution aims to minimize the total area of the phase boundaries as a way to minimize contact between different phases which energetically is not favourable. As such, smaller phase regions tend to shrink or merge with larger domains which leads to a much coarser structure or pattern. This second stage is called Oswald ripening.

Depending on the concrete application, different mathematical models have been developed to model such behaviour. In material science, common models are the Allen-Cahn [AC79] and Cahn-Hilliard equation [Cah61, CH58], which are based on the Ginzburg-Landau energy, or the Ohta-Kawasaki equation [OK86], which is derived from an additional non-local contribution to the Ginzburg-Landau energy. There are many articles discussing the derivation and properties of these models. We refer the reader to [Che02, Che04, CHL10, BHC93] in the case of the Allen-Cahn equation and to [CR03, ONIM99, RW03] in the case of the Ohta-Kawasaki equations. For this thesis, the Cahn-Hilliard equation is the most important example and we refer to the overview [NC08] and to [ES86, Tem97, NST89, NST87] for results on existence as well as long-time behaviour.

This thesis is concerned with the mathematical analysis of a model describing phase separation on biological membranes, in contrast to the above examples which are motivated by problems from material science.

Biological membranes generally consist of bilayers of phospholipid molecules, but can also include other molecules such as cholesterols or proteins. Phospholipids are molecules composed of two hydrophobic fatty acids which are linked through a hydrophilic phosphate group. Due to the physical shape of the lipid molecule, the fatty acids and the phosphate group are usually referred to as tails and head respectively. They arrange themselves in a bilayer, i.e. in two layers of lipid molecules with the hydrophobic tails pointing towards each other.

In eukaryotic cells, such a bilayer cell membrane encloses the cytosol, the cellular fluid inside the cell.

Cell membranes are highly heterogeneous, containing lipids with either saturated or unsaturated tails as well as cholesterols, proteins and other molecules. The lateral organisation of these different components is important for the functioning of the cell, contributing to protein trafficking, endocytosis, and signalling [FSH10b, RL11].

A lot of attention in this context is given to the emergence of so-called lipid rafts. These rafts are intermediate sized domains (10 – 200 nm), characterized as regions consisting mainly of saturated lipid molecules enriched with cholesterols [Pik06]. We refer the reader to the overview [Sch17] and the list of references therein for a discussion of the experimental evidence for their existence.

Due to their structure with a semirigid tail, cholesterol molecules have a strong affinity for saturated lipids, and regions with a high concentration of saturated lipids, which are enriched in cholesterol are much more ordered than regions in which cholesterol is absent [RPGVK09]. During the formation of lipid rafts, such a liquid-ordered phase (l_o, saturated lipids and cholesterols) separates from a liquid-disordered phase (l_d, mostly unsaturated lipids).

However, the evolution of these lipid rafts differs from the coarsening process observed in other examples of phase separation. Instead of merging domains in such a way that the phase boundary is minimized as expected during the Oswald ripening, lipid rafts in biomembranes develop into several finite-size domains.

It has been argued that cell membranes are affected by active cellular processes which contribute to this behaviour and effectively keep the phase separation process from reaching its equilibrium [FSH10a, RL11, GSR08, For05]. As such, active transport processes of membrane components like cholesterol and lipids must be taken into account as non-equilibrium contributions when discussing lipid raft formation from a thermodynamical point of view. In particular, it has been observed that the formation of lipid rafts is linked to the presence of cholesterols in the membrane [LPC⁺13].

Based on this assumption, several theoretical models for the formation of lipid rafts have been proposed, many of them falling into the category of so-called phase-field models.

Phase-field models are an important class of models for phase separation. In particular, the aforementioned Allen-Cahn equation, Cahn-Hilliard equation, and Ohta-Kawasaki equation belong to this category.

In the case of a binary mixture, phase field models introduce an order parameter φ on a domain $\Omega \subset \mathbb{R}^n$ which (up to rescaling) takes values between -1 and 1. The pure phases are represented by domains in which $\varphi = 1$ or $\varphi = -1$ respectively. An important aspect of these models is that they allow φ to take values in the open interval $(-1, 1)$. These values do not correspond to a pure phase. As such, the sharp boundary between phases in these models is replaced by a transition layer, in which the order parameter φ rapidly goes from -1 to 1. As the width of the transition layer goes to zero, the equations approach their sharp-interface limit, which allows only for pure phases $\varphi = \pm 1$ and features explicit evolution equations for the dynamics for the phase interface. In the sharp-interface limit, one thus recovers a geometric evolution equation for the interface motion which might be coupled to partial differential equations in the phase regions.

In the two important examples of the Allen-Cahn equation and the Cahn-Hilliard equation mentioned above, the equations are derived as L^2 and H^{-1}-gradient flows of the Ginzburg-Landau energy

$$\mathcal{F}_\varepsilon(\varphi) = \int_\Omega \frac{\varepsilon}{2} |\nabla\varphi(x)|^2 + \frac{1}{\varepsilon} W(\varphi(x)) \; dx,$$

respectively.

Because of the first term, the Ginzburg-Landau energy favours spatially homogeneous regions. The function W in the second term is usually chosen in such a way that it attains its minima for $\varphi = \pm 1$, making sure that it is energetically optimal for the minimizers to take values which correspond to the different phases. Since the transition layer between different phases corresponds to the transition from $\varphi = -1$ to $\varphi = +1$, it is roughly described by a neighbourhood of the level set $\{\varphi = 0\}$. Choosing W in such a way that it has a local maximum for $\varphi = 0$ thus penalizes the length of the interface and effectively acts as the driving force behind the coarsening described above. Common choices for W include

$$W(s) = (s^2 - 1)^2, \qquad \text{(double-well potential)}$$

$$W(s) = \frac{1}{2}\left[(1+s)\ln(1+s) + (1-s)\ln(1-s)\right] - \frac{1}{2}s^2, \qquad \text{(logarithmic potential)}$$

$$W(s) = \begin{cases} \frac{1}{2}(1-s^2), & s \in [-1,1] \\ +\infty & \text{else.} \end{cases} \qquad \text{(obstacle potential)}$$

Existing phase-field models for lipid raft formation use evolution equations derived from the Ginzburg-Landau energy and add additional terms to account for various non-equilibrium contributions. We refer the reader to the overview articles [Sch17, RL11, FSH10a] for a detailled discussion.

In [For05], Foret proposed a system that models phase separation in a binary mixture (saturated and unsaturated lipids) and includes an exchange of lipid molecules with an external reservoir that only depends on the membrane lipid concentration. While the motivation for this model was the inclusion of non-equilibrium effects, it actually coincides with the Ohta-Kawasaki equations which arises as an equilibrium model in the simulation of diblock copolymers, see [Sch17, Appendix A].

Gómez, Sagués, and Reigada [GSR08] extended the ideas in [For05] to a ternary mixture by including the cholesterol concentration in the membrane as an additional order parameter in the model. As a result, their energy includes an additional term to account for the strong affinity between cholesterols and saturated lipids. Instead of a direct influx of lipids, they consider exchange dynamics for the cholesterol.

In [GKRR16] Garcke, Rätz, Röger and the author extended the model by Gómez, Sagués, and Reigada to also include the cholesterol dynamics in the cytosol. Based on thermodynamic conservation laws, they state evolution equations for the lipid concentration and the cholesterol concentration both in the membrane and in the cytosol. The equations are then coupled by an exchange term for the cholesterols, which is treated as an external source term in both equations.

Different approaches were given by Fan, Sammalkorpi, and Haataja [FSH10b], who model the exchange as stochastic noise and in addition consider short-range interaction between the lipid bilayer and the cytoskeleton (a protein based structure inside eukaryotic cells), and Turner, Sens, and Socci [TSS05]. The last model does not constitute a phase-field model. Instead, it describes the evolution of clusters containing n−molecules. Clusters are allowed to merge or to break into smaller ones. Similar to the approaches before, an additional source term in the model accounts for lipid recycling.

Apart from these models which focus on non-equilibrium effects because of the interaction dynamics between the cell membrane and its environment, there have been several proposals which consider properties of the membrane or its chemical configuration as (additional) driving forces behind the formation of lipid rafts. Aspects taken into account in these models include the possible presence of surfactants in the membrane or a coupling between certain (lipid) concentrations and spontaneous curvature of the membrane. We refer the reader to [Sch17] for a comprehensive introduction to these models and mechanisms.

To the knowledge of the author, recent contributions emphasize the derivation of models and qualitative behaviour or simulations while neglecting other aspects of a detailed mathematical analysis. For the lipid raft model proposed in [GKRR16], we will carry out such an analysis of its mathematical properties in this thesis. Our focus will be the asymptotic behaviour if the influence of the affinity between cholesterols and saturated lipids in the model becomes large, the reduction of the model for infinite cytosolic diffusion and the sharp-interface limit.

Sharp interface limits for the Cahn-Hilliard and Allen-Cahn equation have been extensively studied [ABC94, AHM08, BR93, Che96, CHL10, CENC96, Ilm93]. They correspond to the limit $\varepsilon \searrow 0$, where ε is the parameter in the definition of the Ginzburg-Landau energy. For the Cahn-Hilliard equation, Pego [Peg89] derived formally that the sharp-interface limit for $\varepsilon \searrow 0$ is given by the Mullins-Sekerka equations.

The Mullins-Sekerka equations describe the evolution over a time interval $[0, T]$ of a hypersurface γ_t in $\Omega \subset \mathbb{R}^n$. The normal velocity $\mathcal{V}$ of γ_t is given by

$$\mathcal{V} = [\nabla \mu]_-^+ ,$$

where $[\cdot]_-^+$ is the jump across the interface γ and ν_{γ_t} denotes the unit normal vector to γ_t. The function μ is the solution to

$$\begin{aligned} \Delta\mu &= 0 \quad \text{in } \Omega \setminus \gamma_t, \\ \mu &= \kappa \quad \text{on } \gamma_t, \end{aligned}$$

where κ denotes the mean curvature of the interface γ_t. The boundary condition $\mu = \kappa$ is also called Gibbs-Thomson law.

Pego's result is based on formal matched asymptotic expansions. Rigorous results based on this technique were obtained by Alikakos, Bates, and Chen [ABC94] under the assumption that smooth solutions to the sharp interface problem exist.

Chen gave a suitable definition of a weak solution to the sharp-interface limit of the Cahn-Hilliard equation [Che96] and proved that solutions to the Cahn-Hilliard equation converge to such solutions as $\varepsilon \searrow 0$. His approach is based on tools from geometric measure theory. Instead of treating the phase boundaries as smooth hypersurfaces, he formulates the equations with the help of varifolds, which generalize the notion of a hypersurface. Their origin lies in Plateau's Problem where one seeks to minimize the area of a surface with a given boundary [Alm66].

The main advantage behind the use of varifolds in the context of the Cahn-Hilliard equation is that they allow for the treatment of phantom interfaces, which will naturally occur because phase domains merge or vanish as the coarsening advances. As a result, interfaces may no longer separate two distinct phases and thus become hidden or phantom interfaces. For smooth hypersurfaces, this would result in singularities in the evolution of the sharp interfaces.

One of the main difficulties in the formulation of weak solutions to the Mullins-Sekerka equations is the Gibbs-Thomson law $\kappa = \mu$ on the interface. Since the interface in the weak context can develop kinks in which the classical mean curvature explodes, a weak formulation of the Gibbs-Thomson law requires that a suitable generalized definition of the mean curvature κ is given. Such a generalized mean curvature can be introduced for varifolds, which makes them a suitable tool in this context.

Besides the varifold solutions introduced by Chen for the sharp-interface limit of the Cahn-Hilliard equation, there are also solution concepts based on varifolds for other geometric problems. One example is Brakke's formulation of the mean curvature flow, [Bra78], which occurs as the sharp-interface limit of the Allen-Cahn equation [Ilm93].

For the discussion of the sharp-interface limit in this thesis, we will state the corresponding equations in the varifold formulation introduced by Chen in [Che96], i.e. we adopt his approach.

Apart from the weak solutions to the Mullins-Sekerka model obtained via the sharp-interface limit in the Cahn-Hilliard equation, Luckhaus and Sturzenhecker [LS95] directly constructed weak solutions to the Mullins-Sekerka model via a time-discrete approximation scheme based on functions of bounded variations. Here, the interface is given as the boundary of a set of finite perimeter which describes one of the phase regions. However, their result relies on the assumption that the total area of the interfaces is conserved throughout the limit process. Indeed, Schätzle [Sch97] constructed an example in which the BV-formulation of the Gibbs-Thomson law used in [LS95] breaks down if the total interfacial area is not conserved and phantom interfaces occur.

To remedy this situation, Röger [Rög04] studied the surface area measures associated with the boundary of sets of finite perimeters. Based on this study, he gave a definition of the generalized mean curvature of a set of finite perimeter. Together with a convergence result by Schätzle in [Sch01], this definition allows the construction of solutions to the Gibbs-Thomson law via a time discrete approximation scheme. We will use this approach in the last chapter of this thesis to directly prove the existence of solutions to the sharp-interface limit of the lipid raft model in [GKRR16].

Structure of this Thesis

In Chapter 2 we introduce the model for lipid raft formation derived in [GKRR16]. For the sake of completeness, we recall the modelling process that leads to the derivation of the model from thermodynamics. Based on different assumptions on the constitutive relation behind the cholesterol exchange, we identify so-called equilibrium and non-equilibrium versions of the model and discuss their qualitative behaviour. In particular, we review the numerical simulations carried out in [GKRR16]. These simulations show that the formation of raft-like structures can only be observed for the non-equilibrium versions of the model. Moreover, almost stationary states from the numerical experiments seem to be closely related to stationary states of the so called Ohta-Kawasaki equations.

Since the cytosolic diffusion in the model is usually much larger than the lateral diffusion in the membrane, we furthermore investigate the reduction of the full model for infinite cytosolic diffusion. The corresponding reduced model is formally identified and various reformulations are discussed.

Chapter 3 introduces the necessary mathematical tools for the mathematical analysis of the model in this thesis. We give a quick overview of the involved function spaces before introducing some basic concepts from geometric measure theory which will be used in the discussion of the sharp-interface limit $\varepsilon \searrow 0$. This section in particular includes the definition of a varifold as a generalized surface, the definition of its curvature and the notion of a generalized mean curvature to the boundary of a Caccioppoli set.

The existence and regularity of solutions to the full model is discussed in Chapter 4. Chapter 4 also includes a rigorous discussion of the relation between the full and reduced model as derived in Chapter 2. We remark that the energy estimate proved in this chapter as part of the existence result will find several applications throughout this thesis.

In Chapter 5 and Chapter 6 we investigate some properties of the reduced model. Chapter 5 is concerned with the long time existence of solutions as well as the existence of stationary solutions. The relation between the reduced model and the Ohta-Kawasaki equations is studied in Chapter 6. We prove that a modified Ohta-Kawasaki system can be recovered as a limit from the reduced lipid raft model which helps to explain the numerical experiments in [GKRR16].

The final three chapters are concerned with the sharp-interface limit $\varepsilon \searrow 0$ of the lipid raft model. In this limit, the phase field model derived in [GKRR16] becomes a free boundary problem on the cell membrane coupled to a diffusion equation in the cell. The limit problem was already identified in [GKRR16] via formal asymptotics. For the convenience of the reader, we give a sketch of these arguments in Chapter 7.

The main result related to the sharp-interface limit is the rigorous convergence result in Chapter 8, which is comparable to the corresponding convergence result for solution to the Cahn-Hilliard equation in $\Omega \subset \mathbb{R}^n$ by Chen [Che96]. We introduce varifold solutions to the sharp interface problem and prove that solutions to the phase field based lipid raft model converge to varifold solutions to the sharp interface problem.

Finally, we directly prove the existence of solutions to the sharp interface problem using a time discrete approximation scheme introduced by Röger [Rög04], see Chapter 9.

CHAPTER 2

A Lipid Raft Model Including Cytosolic Diffusion and Cholesterol Exchange

In [GKRR16] Garcke, Rätz, Röger and the author proposed a model for lipid raft formation based on the interplay between a thermodynamic equilibrium process and nonequilibrium effects, in particular active transport processes on the cell membrane. The model is derived from thermodynamic conservation laws, both on the membrane and in the cytosol. The former describes the phase separation between saturated and unsaturated lipid molecules, from which the lipid rafts emerge. The latter describes the dynamic inside the cytosol. The equations on the membrane and in the cytosol are then coupled by an in-/out-flux q related to exchange processes between the cell and its membrane. From the viewpoint of thermodynamics, this exchange term can be interpreted as an external source term in both the membrane and cytosol equations.

The discussion in [GKRR16] shows that the model is thermodynamically consistent for arbitrary constitutive choices for the exchange term q. Moreover, numerical simulations carried out in [GKRR16, Section 5] illustrate how different constitutive choices for q influence the qualitative behaviour of the coupled system. In particular, micro domains (or lipid rafts) do not emerge if the constitutive choices for q imply a decreasing global free energy. This observation justifies the notion that the formation of lipid rafts is a result of non-equilibrium effects influencing the phase separation on the cell membrane.

In this chapter, we introduce the lipid raft model from [GKRR16]. We give a brief review of its derivation from thermodynamics and discuss different choices for the exchange term q. Finally, we present some variants of the lipid raft model. Our aim is to give a short introduction to the lipid raft model. For a comprehensive discussion, we refer the reader to [GKRR16].

2.1 Overview

Lipid rafts are characterized as liquid-ordered phases consisting of saturated lipid molecules. They are enriched of cholesterol and various proteins. Due to the affinity of cholesterol for saturated lipids over unsaturated lipid molecules, Gómez, Sagués, and Reigada [GSR08] proposed a model based on an energy that depends not only on the relative concentration of saturated lipids but also includes the cholesterol concentration as an additional variable. Their energy then features a Ginzburg-Landau type contribution that accounts for the phase separation between saturated and unsaturated lipids, and a second term that models cholesterol-lipid interactions. A source term in the corresponding evolution equations is then used to include cholesterol fluxes between the membrane and its surroundings.

The model proposed in [GKRR16] extends this ansatz. It studies in addition the (simplified) dynamics of cytosolic cholesterol and allows for more involved choices regarding the cholesterol exchange.

Before discussing the derivation of the model from thermodynamics, we give an introductory description of the model. Let $B \subset \mathbb{R}^3$ be a bounded open set with smooth boundary $\Gamma := \partial B$. The set B and the surface Γ represent the cell and its outer membrane respectively. The basic quantities in the model are the rescaled relative concentration φ of saturated lipids in the membrane, the relative concentration v of membrane-bound cholesterol and the relative concentration u of cytosolic cholesterol. We normalize φ such that $\varphi = 1$ represents the pure saturated lipid phase and $\varphi = -1$ within the pure unsaturated lipid phase. Moreover, $v = 1$ and $u = 1$ correspond to maximal saturation for the cholesterol concentrations.

Let now

$$\mathcal{F}(v,\varphi) = \int_\Gamma \frac{\varepsilon}{2}|\nabla\varphi|^2 + \varepsilon^{-1}W(\varphi) + \frac{1}{2\delta}(2v - 1 - \varphi)^2, \tag{2.1}$$

with the double-well potential $W(s) = (1-s^2)^2$. The functional $\mathcal{F}$ consists of two parts. The first part $\int_\Gamma \frac{\varepsilon}{2}|\nabla\varphi|^2 + \varepsilon^{-1}W(\varphi)$ is a classical Ginzburg-Landau energy, modeling the phase separation between the two lipid phases. The second part $\frac{1}{2\delta}(2v-1-\varphi)^2$ accounts for the affinity between saturated lipid molecules and membrane-bound cholesterol.

We now assume that the evolution of the membrane quantities is driven by chemical potentials derived from the functional $\mathcal{F}$. Namely, we introduce

$$\begin{aligned}
\mu &:= \frac{\delta\mathcal{F}}{\delta\varphi} = -\varepsilon\Delta_\Gamma\varphi + \varepsilon^{-1}W'(\varphi) - \delta^{-1}(2v-1-\varphi),\\
\theta &:= \frac{\delta\mathcal{F}}{\delta v} = \frac{2}{\delta}(2v-1-\varphi),
\end{aligned}$$

and say that $\mathcal{F}$ is the surface free energy functional of the model.

We then consider the following bulk–surface system consisting of a surface Cahn–Hilliard equation coupled to a bulk–diffusion equation,

$$\begin{aligned}
\partial_t u &= D\Delta u && \text{in } B\times(0,T], && (2.2)\\
-D\nabla u\cdot\nu &= q && \text{on } \Gamma\times(0,T], && (2.3)\\
\partial_t\varphi &= \Delta_\Gamma\mu && \text{on } \Gamma\times(0,T], && (2.4)\\
\mu &= -\varepsilon\Delta_\Gamma\varphi + \varepsilon^{-1}W'(\varphi) - \delta^{-1}(2v-1-\varphi) && \text{on } \Gamma\times(0,T], && (2.5)\\
\partial_t v &= \Delta_\Gamma\theta + q = \frac{4}{\delta}\Delta_\Gamma v - \frac{2}{\delta}\Delta_\Gamma\varphi + q && \text{on } \Gamma\times(0,T] && (2.6)\\
\theta &= \frac{2}{\delta}(2v-1-\varphi) && \text{on } \Gamma\times(0,T] && (2.7)
\end{aligned}$$

with initial conditions for u, φ and v. Here we denote by ν the outer unit normal vector of B on Γ.

A few comments on the basic ideas included in these equations are in order. From a thermodynamical viewpoint, (2.4) and (2.6) are mass balance equations for the surface quantities. Equations (2.2) and (2.3) model the evolution of the cytosolic cholesterol by a simple diffusion equation, in which the parameter D occurs as the diffusion coefficient. We shall illuminate this further in Remark 2.2. The important part is the inclusion of Neumann boundary conditions for the cytosolic diffusion. Depending on the characterization of the exchange term q, the cholesterol flux from the cytosol B onto the membrane Γ appears as a source term for the evolution of the membrane-bound cholesterol v in Equation (2.6). Equation (2.6) also includes a cross-diffusion, which stems from the cholesterol-lipid affinity in the surface energy $\mathcal{F}$. Finally, Equations (2.4) and (2.5) constitute Cahn-Hilliard dynamics for the lipid concentration and allow for a contribution from the cholesterol evolution via the last term. We note that the parameter δ effectively controls how much the preferred binding between saturated lipids and cholesterols influences the system.

Remark 2.1. Equation (2.4) implies that the total mass of surface lipids is constant in time. Similarly, equations (2.2), (2.3), and (2.6) yield that the combined total mass of surface and cytosolic cholesterol is conserved. We will always denote the total lipid and cholesterol mass by m and M respectively, i.e. for all times

$$\int_\Gamma \varphi \; d\mathcal{H}^2 = m, \qquad \int_B u \; dx + \int_\Gamma v \; d\mathcal{H}^2 = M.$$

We will address different constitutive choices for the exchange term q in Section 2.3 below. For the moment, we only remark that there are two different approaches behind possible choices for q.

One possibility is to choose q in such a way that the global free energy of the coupled membrane/cytosol system is decreasing, i.e.

$$\frac{d}{dt}\left(\mathcal{F}(v(\cdot,t),\varphi(\cdot,t)) + \frac{1}{2}\int_B u(\cdot,t)^2\right) \leq 0.$$

As such, the evolution can be expected to tend to an equilibrium of the free energy $\mathcal{F}$. In [GKRR16] it was observed that such a choice leads to a macro-scale phase separation displaying one connected domain of saturated lipids.

In contrast to this first approach, there are choices for q for which the global free energy is not decreasing, since

$$\frac{d}{dt}\left(\mathcal{F}(v(\cdot,t),\varphi(\cdot,t)) + \frac{1}{2}\int_B u(\cdot,t)^2\right) \leq \int_\Gamma q(\theta - u) \; d\mathcal{H}^2$$

and the right-hand side does not need to be non-positive. In this case, the corresponding evolution can lead to the formation of complex phases, as it is evident by numerical simulations in [GKRR16, Section 5].

2.2 Thermodynamical Justification

In [GKRR16] the model was shown to be thermodynamically consistent. The authors derive the model from mass balance equations for the relative lipid concentration φ and the cholesterol concentration v on the surface Γ as well as the mass balance for the cytosolic cholesterol concentration u. In both the cholesterol mass balance equation on Γ and the cholesterol mass balance equation in B, an external source term is allowed. This yields equations governing the evolution of φ and v on the surface Γ and an evolution equation for u in B. These equations are subsequently coupled by matching the external source terms in the surface and the bulk equations.

In addition to quantities φ, v and u we introduced earlier, we now also consider the mass flux J_v of the membrane bound cholesterol, the mass flux J_u of the bulk cholesterol and the mass flux J_φ of the lipid molecules. Moreover, q denotes an external mass supply of cholesterol on the surface while we denote by q_u an external influx of cholesterol into the bulk. Note that u and J_u are quantities defined in B, in contrast to J_v, J_φ and q that are defined on the surface Γ. As it is the influx into the bulk, q_u is also a quantity on $\Gamma = \partial B$.

Let us now first consider the equations on the surface. For any arbitrary domain $\Sigma \subset \Gamma$ with outer unit conormal ν_Σ, the mass balance equation for the lipid molecules is

$$\frac{d}{dt}\int_\Sigma \varphi \ d\mathcal{H}^2 = -\int_{\partial\Sigma} J_\varphi \cdot \nu_\Sigma \ d\mathcal{H}^1.$$

On the same domain, the mass balance for the surface cholesterol, including the source term q, reads

$$\frac{d}{dt}\int_\Sigma v \ d\mathcal{H}^2 = -\int_{\partial\Sigma} J_v \cdot \nu_\Sigma \ d\mathcal{H}^1 + \int_\Sigma q \ d\mathcal{H}^2.$$

Via the Gauss-Theorem, both equations can be formulated in their local form

$$\partial_t\varphi + \operatorname{div}_\Gamma J_\varphi = 0 \quad \text{on } \Gamma \times (0,T], \tag{2.8}$$

$$\partial_t v + \operatorname{div}_\Gamma J_v = q \quad \text{on } \Gamma \times (0,T]. \tag{2.9}$$

For the equations to be thermodynamically consistent, we require the second law of thermodynamics to hold. In the following, f will denote the surface free energy. In models for phase transitions, the free energy usually depends not only on the order parameter φ and the remaining surface quantities but also on $\nabla_\Gamma\varphi$ (see e.g. [Gur96]) and we shall allow this dependence here as well, i.e. $f = f(\varphi, \nabla_\Gamma\varphi, v)$. Partial derivatives of f with respect to these variables will be denoted by $f_{,x}$, where $x = \varphi, \nabla_\Gamma\varphi$ or v. Furthermore, we denote the chemical potential for the lipid mass evolution by μ and the chemical potential behind the surface cholesterol evolution by θ. Then isothermal formulation of the second law reads

$$\frac{d}{dt}\int_\Sigma f \ d\mathcal{H}^2 \le -\int_{\partial\Sigma} \left(\mu J_\varphi \cdot \nu_\Sigma - (\partial_t\varphi f_{,\nabla_\Gamma\varphi} \cdot \nu_\Sigma) + \theta J_v \cdot \nu_\Sigma\right) \ d\mathcal{H}^1 + \int_\Sigma \theta q \ d\mathcal{H}^2$$

or equivalently

$$\partial_t f + \operatorname{div}_\Gamma\left(\mu J_\varphi - \partial_t\varphi f_{,\nabla_\Gamma\varphi} + \theta J_v\right) \le \theta q \quad \text{on } \Gamma \times (0,T] \tag{2.10}$$

in the local form.

Since our aim here is an introductory exposition, we refer the reader to [GKRR16, Section 2] and the references therein for a discussion about how equation (2.10) fits into the more general framework of thermodynamical modeling of phase separation processes. We only state that the particular form of the second law of thermodynamics in (2.10) is due to the fact that the free energy f is allowed to depend on $\nabla_\Gamma\varphi$. As such, (2.10) also includes the term $\partial_t\varphi f_{,\nabla_\Gamma\varphi}$.

With the constitutive relation

$$f = f(\varphi, \nabla_\Gamma\varphi, v)$$

we deduce from (2.10) the free energy inequality

$$f_{,\varphi}\partial_t\varphi + f_{,v}\partial_t v \nabla_\Gamma + \mu \cdot J_\varphi + \nabla_\Gamma\theta \cdot J_v + (\operatorname{div}_\Gamma J_\varphi)\,\mu + (\operatorname{div}_\Gamma J_v)\,\theta - \partial_t\varphi \operatorname{div}_\Gamma f_{,\nabla_\Gamma\varphi} \le \theta q.$$

Using Equations (2.8) and (2.9), we infer

$$\left(f_{,\varphi} - \operatorname{div}_\Gamma f_{,\nabla_\Gamma\varphi} - \mu\right)\partial_t\varphi + \left(f_{,v} - \theta\right)\partial_t v + \nabla_\Gamma\mu \cdot J_\varphi + \nabla_\Gamma\theta \cdot J_v + \le 0. \tag{2.11}$$

We now interpret (2.11) as a constraint on the constitutive relations. In general, solutions to the balance equations can attain arbitrary values for $\partial_t\varphi$ and $\partial_t v$. Hence in order for the left hand-side in (2.11) to be non-positive, the factors $(f_{,\varphi} - \operatorname{div}_\Gamma f_{,\nabla_\Gamma\varphi} - \mu)$ and $(f_{,v} - \theta)$ have to vanish. This is Liu's method of Lagrange multipliers and we refer the reader to [Liu02] for more details.

One option to make sure that (2.11) holds for all states of the system (2.8)–(2.9) is to choose

$$\mu = f_{,\varphi} - \operatorname{div}_\Gamma(f_{,\nabla_\Gamma\varphi}), \tag{2.12}$$
$$\theta = f_{,v}, \tag{2.13}$$
$$J_\varphi = -D_\varphi\nabla_\Gamma\mu, \tag{2.14}$$
$$J_v = -D_v\nabla_\Gamma\theta, \tag{2.15}$$

where $D_\varphi, D_v \geq 0$.

We now turn our attention to the governing equations for the cholesterol evolution inside the bulk B. We proceed similarly as before and consider the mass balance for all domains $U \subset B$. As before, we add a source term q_u to the mass balance. In contrast to the surface equations, this source term models an influx through the membrane into the bulk and is therefore only defined on $\Gamma = \partial B$. With these considerations, the mass balance equation reads

$$\frac{d}{dt}\int_U u\ dx = -\int_{(\partial U)\setminus\Gamma} J_u\cdot\nu_U\ d\mathcal{H}^2 + \int_{(\partial U)\cap\Gamma} q_u\ d\mathcal{H}^2.$$

For the bulk chemical potential μ_u and the free energy $f_b = f_b(u)$, the energy inequality

$$\frac{d}{dt}\int_U f_b(u)\ dx \leq -\int_{(\partial U)\setminus\Gamma} \mu_u J_u\cdot\nu_U\ d\mathcal{H}^2 + \int_{(\partial U)\cap\Gamma} \mu_u q_u\ d\mathcal{H}^2$$

has to hold in order to make the equations thermodynamically consistent. We derive again local forms of these equations by looking at all $U \subset B$ such that $\partial U \cap \Gamma = \varnothing$ and find

$$\partial_t u + \operatorname{div} J_u = 0 \quad \text{on } B\times(0,T], \tag{2.16}$$
$$\partial_t f_b(u) + \operatorname{div}(\mu_u J_u) \leq 0 \quad \text{on } B\times(0,T]. \tag{2.17}$$

Note that the source term q_u does not appear on the right hand-side in (2.17) since it is only defined on Γ. If we consider $U \subset B$ such that $\partial U \cap \Gamma \neq \varnothing$ the mass balance yields

$$0 = \int_U \left(\partial_t u + \operatorname{div} J_u\right)\ dx = \int_{\partial U\cap\Gamma} \left(q_u + J_u\cdot\nu_\Gamma\right)\ d\mathcal{H}^2$$

and we infer the boundary condition

$$q_u = -J_u\cdot\nu_\Gamma \quad \text{in } \Gamma\times(0,T]. \tag{2.18}$$

Again, the equations (2.16)–(2.18) have to hold for all possible states of the system. In particular, the choice

$$\mu_u = f_b'(u) \tag{2.19}$$
$$J_u = -M(u)\nabla\left(f_b'(u)\right) \tag{2.20}$$

is sufficient to make sure that this is true for (2.17). Here the function $M : \mathbb{R}\to\mathbb{R}$ is the mobility of u.

Remark 2.2. Equations (2.12)–(2.15) and (2.19)–(2.19) show that in order to derive a system of specific equations, it is sufficient to make specific choices for the surface free energy density f, the bulk free energy density f_u, the mobility M and the source terms q and q_u. Our discussion so far has been independent of these choices for f, f_u, M and in particular the source terms q and q_u, showing that we arrive in all cases at a thermodynamically consistent model.

The system (2.2)–(2.7) can be derived from the foregoing discussion if we choose for $\varepsilon, \delta > 0$ and a bulk diffusion constant $D > 0$

$$\begin{aligned} f(\varphi, \nabla_\Gamma \varphi, v) &= \frac{\varepsilon}{2} |\nabla_\Gamma \varphi|^2 + \frac{1}{\varepsilon} W(\varphi) + \frac{1}{2\delta} (2v - 1 - \varphi)^2, \\ f_b(u) &= \frac{1}{2} u^2, \\ M(u) &= \frac{D}{f_b''(u)} \end{aligned}$$

and set $D_\varphi = D_v = 1$ in (2.14) and (2.15) respectively. Again we choose $W(s) = (1 - s^2)^2$.

Thus the above choice for the surface free energy density f directly leads to the functional $\mathcal{F}$ defined in (2.1). One readily checks that in this case the relation (2.12) yields (2.5) and that (2.13) results in (2.7). The mass balance equations (2.8) and (2.9) together with the laws for the mass fluxes J_φ and J_v in (2.14) and (2.15) respectively give the surface equations (2.4) and (2.6). Similarly, the mass balance (2.16) in the bulk and the influx condition (2.18) correspond to (2.2) and (2.3). Finally, we couple the bulk and the surface equations by setting $q = q_u$.

2.3 Qualitative Behaviour and Different Choices for the Exchange Term

The motivation behind the model (2.2)–(2.7) was the formation of lipid rafts in biological membranes, i.e. to derive evolution equations that display mesoscale patterns as time evolves. It is thus a natural question to study the qualitative behaviour of the model (2.2)–(2.7). We will therefore quickly summarize the relevant findings in [GKRR16].

Throughout this section, we consider the coupled model. That is, we assume $q = q_u$.

Lemma 2.3 ([GKRR16, Lemma 2.1])**.** Assume that the mass balance equations (2.16),(2.8), and (2.9) hold and that $q = q_u$. Then

$$\frac{d}{dt}\left(\int_B u \; dx + \int_\Gamma v \; d\mathcal{H}^2\right) = 0, \quad \frac{d}{dt}\int_\Gamma \varphi \; d\mathcal{H}^2 = 0.$$

If in addition the free energy inequalities (2.10) and (2.17) are true, we have

$$\frac{d}{dt}\left(\int_\Gamma f(\varphi, \nabla_\Gamma \varphi, v) \; d\mathcal{H}^2 + \int_B f_b(u) \; dx\right) \leq \int_\Gamma q(\theta - u) \; d\mathcal{H}^2. \tag{2.21}$$

Proof. The first equation follows if we integrate (2.16) and (2.9), use the divergence theorem and add the resulting equations. Similarly, Equation (2.8) and the divergence theorem yield the second equation in the lemma. The inequality (2.21) follows from (2.10) and (2.17) if one plugs the specific choices (2.12)–(2.15) and (2.19)–(2.20) into these inequalities. ☐

The first two equations in the above lemma are conservation properties for the lipid molecules and for the combined surface and bulk cholesterol. They generalize Remark 2.1 for the concrete model (2.2)–(2.7).

The inequality (2.21) allows to identify two different classes of constitutive laws for the exchange term q. Every constitutive law that implies that $\int_\Gamma q(\theta - u)$ is non-positive implies that the energy of the coupled system is decreasing. As mentioned before, we expect the evolution in this case to approach an equilibrium of $\mathcal{F}$ as $t \to \infty$. Hence we will refer to choices for q that lead to a decreasing energy as equilibrium cases. One example for such an exchange term is

$$q = -c(\theta - u), \qquad c \geq 0. \tag{2.22}$$

On the other hand, there are choices for q such that $\int_\Gamma q(\theta - u)$ does not need to be non-positive. For systems including such an exchange term q, it is not reasonable to expect the evolution to attain equilibrium points of $\mathcal{F}$ as $t \to \infty$, as it is a priori not even certain that solutions exist for all times or that $\mathcal{F}$ is bounded in time. Hence these systems are called non-equilibrium models.

One possible approach leading to a non-equilibrium model is to see the cholesterol attachment to the membrane as a "reaction" between free sites on the membrane, namely regions of low membrane-bound cholesterol concentration v and the cytosolic cholesterol, whereas the detachment from the membrane can be considered to be proportional to v. This results in the constitutive choice

$$q(u, v) := c_1 u(1 - v) - c_2 v \tag{2.23}$$

with positive constants $c_1, c_2 \in \mathbb{R}$. The numerical simulations in [GKRR16, Section 5] show that this approach leads to the formation of complex phases. As such, the choice (2.23) will be treated as a prime example of a non-equilibrium system throughout this thesis. It turns out that the resulting system displays a surprising relationship to the so-called Ohta-Kawasaki system arising in the modeling of diblock copolymers. We shall investigate this relation further in Chapter 6.

For the reduced model derived in Section 2.4 below, we can further illustrate the differences in the qualitative behaviour between the equilibrium and the non-equilibrium cases. In [GKRR16], the authors considered the two exchange terms (2.22) and (2.23) as model cases for the equilibrium and non-equilibrium situations respectively. The numerical simulations in this reference allow to compare the qualitative behaviour in the two cases.

Figures 2.1 and 2.2 show the evolution of the relative lipid concentration φ in the equilibrium and non-equilibrium case for various time steps. In the equilibrium case, the simulations display the saturated lipids clustered in one connected domain, in contrast to the complex patterns observed in the formation of lipid rafts. On the other hand, the non-equilibrium case pictured in Figure 2.2 exhibits the emergence of patterns similar to the formation of lipid rafts.

This observation can be expected, as is discussed in [GKRR16, Section 3.3]. There the authors provide a formal argument by which minimizers of the surface free energy $\mathcal{F}$ in (2.1) must already be minimizers of the usual Ginzburg-Landau energy. As it is reasonable to expect convergence to local energy minimizers in the equilibrium case, the qualitative behaviour in Figure 2.1 is in line with this argument.

Simulation results for the non-equilibrium case with varying initial data are pictured in Figure 2.3. Depending on the initial value of the lipid concentration φ, the almost stationary solutions obtained from the simulation display two distinct classes of patterns, with stripe like patterns emerging if the concentration of saturated and unsaturated lipids is balanced. For less balanced initial values, the experiments show patterns with several circular domains, similar to lipid rafts.

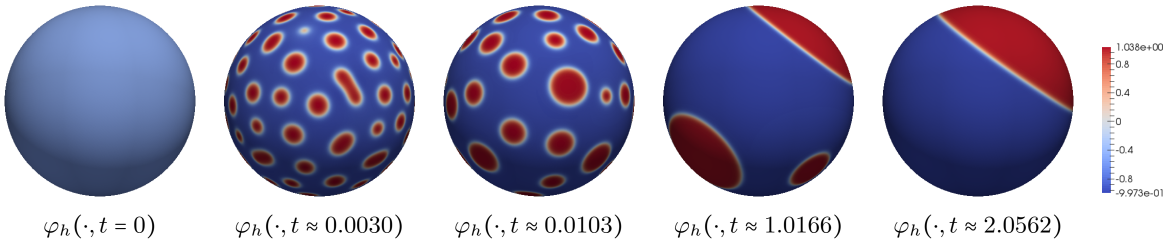


Figure 2.1: Numerical results for the reduced model with a choice of q leading to an energy decreasing evolution and initial data $\varphi(\cdot, 0) = -0.5 + \mathcal{R}$. Here $\mathcal{R} : \Gamma \to [-0.001, 0.001]$ denotes an irregular and nonperiodic oscillation around zero. Contour plots of $\varphi(\cdot, t)$ for several choices of times t. Simulation by A. Rätz, reprinted from [GKRR16].

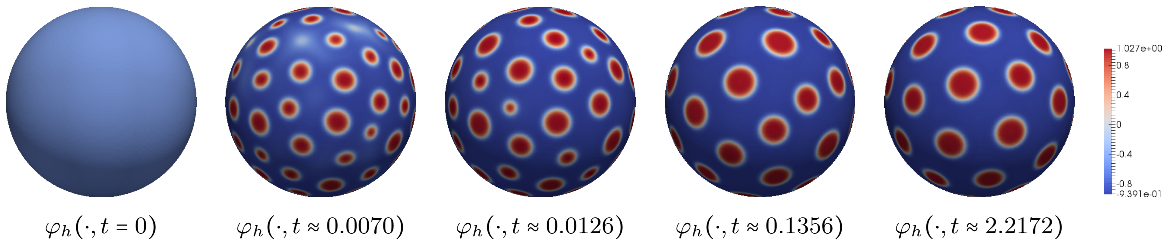
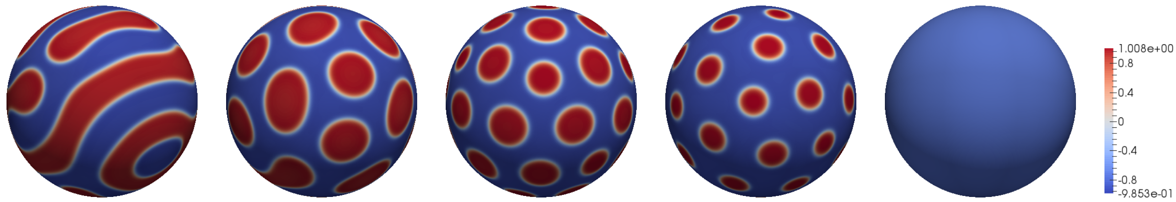


Figure 2.2: Numerical results for diffuse interface approximation of the full model (2.2)–(2.7) with q as in (2.23), leading to a non-decreasing energy. Contour plots of φ for several choices of times t. Simulation by A. Rätz, reprinted from [GKRR16].

A similar behaviour for the stationary states can be seen in the so-called Ohta-Kawasaki equations. As it turns out, the almost stationary states obtained from the simulations of the lipid raft model and the stationary states of the Ohta-Kawasaki equations are closely related as can be seen by the experiment in Figure 2.4. For this experiment, the almost stationary solutions to the reduced system were used as initial data in the Ohta-Kawasaki equations. The corresponding solution to the Ohta-Kawasaki equations attains an almost stationary state which is close to the initial data, i.e. the stationary state of the reduced lipid raft model. As indicated by this simulation, the reduced lipid raft model has indeed a close relation to the Ohta-Kawasaki equations as we shall discuss in Chapter 6.

The stationary states for varying values of δ as pictured in Figure 2.5 provide more insight into this relation. The influence of the preferred binding of the saturated lipid molecules with membrane-bound cholesterol in the energy $\mathcal{F}$ in (2.1) becomes large for small values of δ. Figure 2.5 shows that the number of domains or lipid rafts increases as δ decreases and in particular that for large δ there is no formation of several domains. That is, the qualitative behaviour of the non-equilibrium model for large δ resembles the equilibrium case while for small δ the behaviour of the system seems to be closer to the Ohta-Kawasaki equations. As such it is reasonable to conjecture that as $\delta \to 0$, solutions to the reduced model in the non-equilibrium case $q = c_1 u(1-v) - c_2 v$ should approach solutions to the Ohta-Kawasaki equations. At least for the mean value free parts of the solutions, this behaviour will actually be proved in Proposition 6.1.

We now turn our attention to the growth of $\mathcal{F}$ and allow for a general q that may depend on φ, v, and u. For the moment, we do not distinguish between equilibrium and non-equilibrium



$\hat{\varphi} = 0$ $\hat{\varphi} = -0.1$ $\hat{\varphi} = -0.25$ $\hat{\varphi} = -0.5$ $\hat{\varphi} = -0.75$

Figure 2.3: Almost stationary solutions φ to the reduced system for different initial values $\hat{\varphi}$. Simulation by A. Rätz, reprinted from [GKRR16].

cases. Instead, we show that in both cases $\mathcal{F}$ as defined in (2.1) does not explode in finite time if q grows at most linearly and if we assume the existence of smooth solutions.

Proposition 2.4 ([GKRR16, Proposition 3.1])**.** *Assume that q has at most linear growth, that is there exists $\Lambda > 0$ such that*

$$|q(\varphi, u, v)| \leq \Lambda(1 + |\varphi| + |u| + |v|) \quad \text{for all } \varphi, u, v \in \mathbb{R}. \tag{2.24}$$

Then for all $0 < t < T$ and all $D \geq D_0 > 0$, $0 < \varepsilon \leq \varepsilon_0$ any (smooth) solution to (2.2)–(2.7) *with initial data φ_0, u_0, v_0 satisfies*

$$\mathcal{F}(v(\cdot,t), \varphi(\cdot,t)) + \frac{1}{2}\int_B u(\cdot,t)^2 + \int_0^t \int_B \frac{D}{2}|\nabla u|^2 \leq C(\delta, \Lambda, T, D_0, \varepsilon_0, v_0, \varphi_0, u_0). \tag{2.25}$$

Remark 2.5. Since Proposition 2.4 as it is stated in [GKRR16] assumes the existence of smooth solutions, it should be seen as a formal argument. We will provide a rigorous discussion

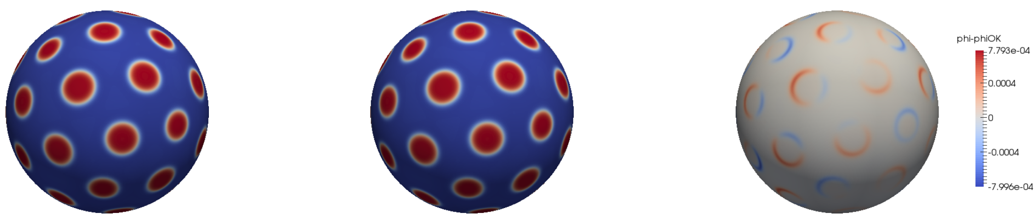
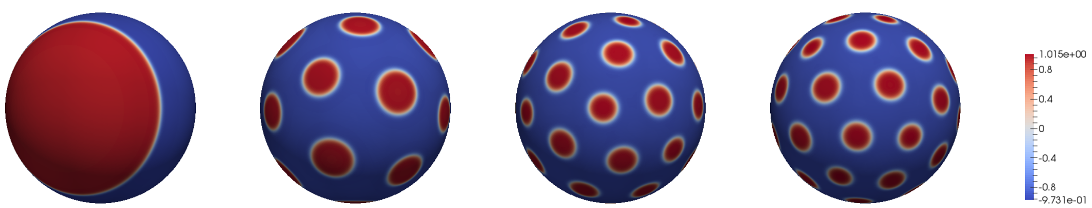


Figure 2.4: Almost stationary φ_h obtained from a simulation of the reduced system (left) and from subsequent Ohta–Kawasaki-based dynamics (middle), difference between the previous numerical solutions (right). Simulation by A. Rätz, reprinted from [GKRR16].



$\delta = 0.3$ $\delta = 0.1$ $\delta = 0.02$ $\delta = 0.002$

Figure 2.5: Almost stationary discrete solutions φ to the reduced model for different values of δ. Simulation by A. Rätz, reprinted from [GKRR16].

in Chapter 4. There we shall prove the existence of weak solutions to the system (2.2)–(2.7), see Theorem 4.2. Moreover, the theorem also asserts that the energy bound (2.25) does hold for these weak solutions to the system (2.2)–(2.7).

Proof of Proposition 2.4. We only give a sketch of the proof since we will present analogue arguments when we prove the rigorous result in Theorem 4.2. Let (u,φ,v,μ,θ) be a (smooth) solution to (2.2)–(2.7) and let $\mathcal{F}$ be as in (2.1). A direct calculation yields

$$\frac{d}{dt}\left(\mathcal{F}(v(\cdot,t),\varphi(\cdot,t))+\frac{1}{2}\int_B u(\cdot,t)\right) = -\int_B D\,|\nabla u(\cdot,t)|^2 - \int_\Gamma \left(|\nabla_\Gamma \mu(\cdot,t)|^2 + |\nabla_\Gamma \theta(\cdot,t)|^2\right)$$
$$-\int_\Gamma (\theta-u)(\cdot,t)q(\varphi(\cdot,t),u(\cdot,t),v(\cdot,t)). \tag{2.26}$$

The growth assumption (2.24) on q implies the estimate

$$\begin{aligned}\left|\int_\Gamma (\theta-u)(\cdot,t)q(\varphi(\cdot,t),u(\cdot,t),v(\cdot,t))\right| &\le \int_\Gamma \left(\theta^2+u^2+C_\Lambda(1+\varphi^2+u^2+v^2)\right)\,d\mathcal{H}^2\\ &= \int_\Gamma \theta^2\,d\mathcal{H}^2 + C_\Lambda\int_\Gamma (1+\varphi^2)\,d\mathcal{H}^2 + (C_\Lambda+1)\int_\Gamma u^2\,d\mathcal{H}^2\\ &\quad + C_\Lambda\int_\Gamma v^2\,d\mathcal{H}^2.\end{aligned}$$

Following the arguments in the proof in [GKRR16], all terms on the right hand-side can be controlled by the energy $\mathcal{F}$ plus a constant. We also refer the reader to the proof of Theorem 4.2, where the arguments will be presented in detail. Hence we deduce

$$\left|\int_\Gamma (\theta-u)(\cdot,t)q(\varphi(\cdot,t),u(\cdot,t),v(\cdot,t))\right| \le C\left(1+\mathcal{F}(v(\cdot,t),\varphi(\cdot,t))+\frac{1}{2}\int_B u(\cdot,t)^2\right)$$

and the claim follows from (2.26) and Gronwall's inequality. □

The above proof shows

$$\left|\int_\Gamma (\theta-u)(\cdot,t)q(\varphi(\cdot,t),u(\cdot,t),v(\cdot,t))\right| \le C\mathcal{F}(v(\cdot,t),\varphi(\cdot,t))+\frac{1}{2}\int_B u(\cdot,t)^2,$$

where the right hand-side is now bounded by (2.25). Hence we deduce from (2.26) the following corollary.

Corollary 2.6. Assume that q has at most linear growth, that is there exists $\Lambda>0$ such that

$$|q(\varphi,u,v)| \le \Lambda(1+|\varphi|+|u|+|v|) \quad \text{for all } \varphi,u,v\in\mathbb{R}.$$

Then for all $0<t<T$ and all $D\ge D_0>0$, $0<\varepsilon\le\varepsilon_0$ any (smooth) solution to (2.2)-(2.7) with initial data φ_0,u_0,v_0 satisfies

$$\mathcal{F}(v(\cdot,t),\varphi(\cdot,t))+\frac{1}{2}\int_B u(\cdot,t)^2+\int_0^t\int_B \frac{D}{2}|\nabla u|^2+\int_0^t\int_\Gamma \left(|\nabla_\Gamma\mu(\cdot,t)|^2+|\nabla_\Gamma\theta(\cdot,t)|^2\right)$$
$$\le C(\delta,\Lambda,T,D_0,\varepsilon_0,v_0,\varphi_0,u_0). \tag{2.27}$$

2.4 A Reduced Model - the Case of Large Cytosolic Diffusion

In the modelling process, the parameter D was the diffusion constant associated with the cytosolic diffusion. This diffusion is often much higher than the lateral diffusion on the cell membrane. The bound in Corollary 2.6 implies that $\|\nabla u\|_{L^2((0,T);L^2(B))} \to 0$ as $D \to \infty$. Hence we expect u to be spatially constant in the limit $D \to \infty$. Thus it is reasonable to view the limit $D \to \infty$ as a reduction of the system (2.2)–(2.7). The aim of this section is to formally characterize the reduced system in the limit $D \to \infty$.

In the resulting system, u is spatially constant and its evolution in time is governed by an ordinary differential equation which is coupled to the surface diffusion for v. If we formally send $D \to \infty$ in (2.2)–(2.7), we derive the system

$$\partial_t u = -\frac{1}{|B|}\int_\Gamma q(\varphi, u, v) \qquad \text{for } t \in (0,T], \tag{2.28}$$

$$\partial_t \varphi = \Delta_\Gamma \mu \qquad \text{on } \Gamma \times (0,T], \tag{2.29}$$

$$\mu = -\varepsilon \Delta_\Gamma \varphi + \varepsilon^{-1} W'(\varphi) - \delta^{-1}(2v - 1 - \varphi) \qquad \text{on } \Gamma \times (0,T], \tag{2.30}$$

$$\partial_t v = \Delta_\Gamma \theta + q(\varphi, u, v) \qquad \text{on } \Gamma \times (0,T], \tag{2.31}$$

$$\theta = \frac{2}{\delta}(2v - 1 - \varphi) \qquad \text{on } \Gamma \times (0,T]. \tag{2.32}$$

Equation (2.29) implies that the total mass $\int_\Gamma \varphi \, d\mathcal{H}^2$ is conserved over time, i.e.

$$\frac{d}{dt}\int_\Gamma \varphi \, d\mathcal{H}^2 = 0. \tag{2.33}$$

Moreover, we infer from (2.28) and (2.31)

$$\frac{d}{dt}\left(\int_B u \, dx + \int_\Gamma v \, d\mathcal{H}^2\right) = 0. \tag{2.34}$$

We remark that (2.28) and (2.34) respectively mean that we reduced the coupled bulk-surface system into a system of surface equations with nonlocal contributions, namely through the characterization of u via either the mass constraint (2.34) or via the integral on the right hand-side of (2.28).

Remark 2.7. Theorem 4.2 will establish Estimate (2.27) rigorously for weak solutions to (2.2)–(2.7). The energy bound then infers a rigorous connection between the full model and the reduced model, justifying the formal considerations above. We refer the reader to Proposition 4.9 for further details.

Remark 2.8. 1. With the discussion on the qualitative behaviour in the non-equilibrium case in mind, we note that the exchange term q as in (2.23) does not fulfil the linear growth condition required in Proposition 2.4. However, we can choose $\tilde{q}$ in such a way that it coincides with (2.23) on a (possibly large) bounded domain and fulfils the growth assumption everywhere else. To this end, let M be as in Remark 2.1 and $\eta : \mathbb{R} \to \mathbb{R}$ fulfil $\eta(s) = s$ for $|s| \le M\,|B|^{-1}$ and assume that η is smooth, monotone increasing and uniformly bounded everywhere else. We now define $\tilde{q}$ as

$$\tilde{q}(u, v) = c_1 u - c_1 \eta(u) v - c_2 v$$

and note that $\tilde{q}$ fulfils the linear growth assumption and coincides with

$$q(u,v) = c_1 u(1-v) - c_2 v = c_1 u - c_1 uv - c_2 v$$

if $0 \le \int_B u(t)\ dx \le M$ for all $t \ge 0$. For the corresponding reduced model, the mass conservation (2.34) reads

$$\int_B u\ dx + \int_\Gamma v\ d\mathcal{H}^2 = M \Leftrightarrow \int_\Gamma v\ d\mathcal{H}^2 = M - \int_B u\ dx.$$

If we consider (2.28) and use the previous equation, we find the specific reformulation

$$\begin{aligned}\frac{d}{dt}\int_B u(t)\ dx &= -\int_\Gamma \tilde{q}(u,v) \\ &= -\frac{c_1|\Gamma|}{|B|}\int_B u(t)\ dx + \left(c_1\eta\left(\frac{1}{|B|}\int_B u(t)\ dx\right) + c_2\right)\left(\int_\Gamma v(t)\ d\mathcal{H}^2\right) \\ &= -\frac{c_1|\Gamma|}{|B|}\int_B u(t)\ dx + \left(c_1\eta\left(\frac{1}{|B|}\int_B u(t)\ dx\right) + c_2\right)\left(M - \int_B u\ dx\right) \qquad (2.35)\end{aligned}$$

of the ordinary differential equation for u. Thus the equation is actually independent of v. The right hand-side is strictly positive if $\int_B u(t)\ dx = 0$ and strictly negative if $\int_B u(t)\ dx = M$. Thus we infer that

$$u(t) \in \left[0, |B|^{-1} M\right] \text{ for all } t \ge 0$$

if the initial data was in this range to begin with. Hence for suitable initial data, we actually have

$$\tilde{q}(u,v) = q(u,v) \text{ for all } t \ge 0.$$

We thus continue to consider the specific form $q(u,v) = c_1 u(1-v) - c_2 v$ in the reduced model as a prime example for the equilibrium case and in particular in Chapter 6.

2. We also infer from the discussion above that the solution u to (2.28) in the reduced model stays bounded for all times if the initial data is in $\left[0, M|B|^{-1}\right]$. Moreover, $u(t) \to u_\infty$ as $t \to \infty$, where

$$u_\infty = \frac{1}{2}\left(\frac{M-|\Gamma|}{|B|} - \frac{c_2}{c_1}\right) + \sqrt{\frac{1}{4}\left(\frac{M-|\Gamma|}{|B|} - \frac{c_2}{c_1}\right)^2 + \frac{c_2 M}{c_1|B|}}$$

is the positive zero of the right-hand side in (2.35).

3. In Sections 5.1 and 5.2 we will not only need to assume that the exchange term q growth at most linearly but will instead assume a stronger growth condition. We assume that there exists $\alpha > 1$ such that

$$|q(u,v)| \le C\left(1 + |u|^{1/\alpha} + |v|^{1/\alpha}\right)$$

for some $C > 0$. A similar argument as before shows that also in these cases we can consider q as in (2.23) if we modify it with suitable cut-off functions.

2.5 A Reformulation for the Reduced Model - Treating the Mean Values Explicitly

Equation (2.28) in the reduced model constitutes an ordinary differential equation for u where the right hand side depends on v and u. Together with the mass constraint (2.34) we have thus a system of two equations in which u and v are the unknown quantities. Moreover, we have that $\frac{d}{dt}\int_\Gamma \varphi = 0$ and the algebraic relation $\int_\Gamma \theta = \frac{2}{\delta}\int_\Gamma (2v - 1 - \varphi)$. This implies that equations (2.29)–(2.32) effectively only govern the evolution of the mean value free functions $\varphi_\Gamma := \varphi - \frac{1}{|\Gamma|}\int_\Gamma \varphi, v_\Gamma := v - \frac{1}{|\Gamma|}\int_\Gamma v$, and $\theta_\Gamma := \theta - \frac{1}{|\Gamma|}\int_\Gamma \theta$, which allows us to decouple the system into a set evolution equations for the mean values and a set of evolution equations for the mean value free parts.

The total mass $\int_B u\, dx + \int_\Gamma v\, d\mathcal{H}^2$ is determined by the initial conditions, since it is constant in time by the mass conservation in (2.34). Again, we denote it by M.

Moreover, we denote by P_Γ the projection onto the mean value free part, i.e. $P_\Gamma f := f - \frac{1}{|\Gamma|}\int_\Gamma f$. Projecting each equation onto its mean value free part, we arrive at

$$\partial_t \varphi_\Gamma = \Delta_\Gamma \mu_\Gamma \qquad \text{on } \Gamma \times (0,T], \tag{2.36}$$

$$\mu_\Gamma = -\varepsilon \Delta_\Gamma \varphi_\Gamma + \varepsilon^{-1} P_\Gamma W'(\varphi) - \frac{\theta_\Gamma}{2} \qquad \text{on } \Gamma \times (0,T], \tag{2.37}$$

$$\partial_t v_\Gamma = \Delta_\Gamma \theta_\Gamma + P_\Gamma q(u,v) \qquad \text{on } \Gamma \times (0,T] \tag{2.38}$$

$$\theta_\Gamma = \frac{2}{\delta}(2v_\Gamma - \varphi_\Gamma) \qquad \text{on } \Gamma \times (0,T] \tag{2.39}$$

together with the equations

$$\frac{d}{dt}\int_B u(t) = -\int_\Gamma q(u,v) \qquad \text{on } (0,T], \tag{2.40}$$

$$\frac{d}{dt}\int_\Gamma \varphi = 0 \qquad \text{on } (0,T], \tag{2.41}$$

$$\int_\Gamma v = M - \int_B u \qquad \text{on } (0,T], \tag{2.42}$$

$$\int_\Gamma \mu = \int_\Gamma \left(\varepsilon^{-1} W'(\varphi) + \frac{\theta}{2}\right) \qquad \text{on } (0,T], \tag{2.43}$$

$$\int_\Gamma \theta = \frac{2}{\delta}\int_\Gamma [2v - 1 - \varphi] \qquad \text{on } (0,T] \tag{2.44}$$

for the mean values.

2.6 A Reformulation for the Reduced Model in the Case $q = c_1 u(1-v) - c_2 v$

In the case that the exchange term q is given in the explicit form $q(u,v) = c_1 u(1-v) - c_2 v$ the numerical simulations in [GKRR16] showed a close relation between the reduced model and the Ohta-Kawasaki equations. As a starting point for the detailed analysis in Chapter 6, we study the previous reformulation of the reduced model in more detail. Our aim is to use the explicit form of the exchange term q and the mass conservation

$$\frac{d}{dt}\left(\int_B u + \int_\Gamma v\right) = 0$$

to derive an explicit ODE for u which does not depend on $\int_\Gamma v$. Furthermore, we substitute $v = \frac{\delta}{4}\theta + \frac{1+\varphi}{2}$ in the equations.

Let $(u, v, \varphi, \mu, \theta)$ be a solution to the reduced model.

Our starting point is equation (2.40). We have $\int_\Gamma v = M - \int_B u$ for all times and thus (2.40) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\int_B u(t) =& - \int_\Gamma q(u(t), v(\cdot, t)) \\ =&(c_1 u(t) + c_2)\left(M - \int_B u(t)\right) - c_1 \frac{|\Gamma|}{|B|}\int_B u(t) \\ =& - \frac{c_1}{|B|}\left(\int_B u(t)\right)^2 + \left(c_1 \frac{M - |\Gamma|}{|B|} - c_2\right)\int_B u(t) + c_2 M\end{aligned}$$

At the same time, we can use equation (2.44) to calculate

$$\begin{aligned}\int_\Gamma q(u(t), v(x,t)) &= c_1 u(t)\int_\Gamma \left(1 - \frac{\delta}{4}\theta - \frac{1+\varphi}{2}\right) - c_2 \int_\Gamma \left(\frac{\delta}{4}\theta + \frac{1+\varphi}{2}\right) \\ &= c_1 u(t) - (c_1 u(t) + c_2)\int_\Gamma \left(\frac{\delta}{4}\theta + \frac{1+\varphi}{2}\right)\end{aligned}$$

where we have used that $v = \frac{\delta}{4}\theta + \frac{1+\varphi}{2}$ almost everywhere. This infers

$$\begin{aligned}P_\Gamma q(u,v) &= q(u,v) - \fint_\Gamma q(u,v) \\ &= -(c_1 u(t) + c_2)\left[\frac{\delta}{4}\left(\theta - \fint_\Gamma \theta\right) + \frac{1}{2}\left(\varphi - \fint_\Gamma \varphi\right)\right].\end{aligned}$$

Thus we can rewrite the equation

$$\partial_t v_\Gamma = \Delta_\Gamma \theta_\Gamma + q_\Gamma$$

to read

$$\frac{\delta}{4}\partial_t \theta_\Gamma = \Delta_\Gamma \theta_\Gamma - \frac{1}{2}\Delta_\Gamma \mu_\Gamma - \frac{\delta\,(c_1 u(t) + c_2)}{4}\theta_\Gamma - \frac{(c_1 u(t) + c_2)}{2}\varphi_\Gamma,$$

effectively eliminating v_Γ from the equation.

Moreover, the equation for $\int_\Gamma \theta$ implies

$$\int_\Gamma v = \frac{\delta}{4}\int_\Gamma \theta + \int_\Gamma \frac{1+\varphi}{2}$$

which we use to eliminate $\int_\Gamma v$ in the second equation for the mean values.

Summing up our findings, we obtain the system

$$\partial_t \varphi_\Gamma = \Delta_\Gamma \mu_\Gamma \quad \text{on } \Gamma \times (0,T], \tag{2.45}$$

$$\mu_\Gamma = -\varepsilon \Delta_\Gamma \varphi_\Gamma + \varepsilon^{-1} P_\Gamma W'(\varphi) - \frac{\theta_\Gamma}{2} \quad \text{on } \Gamma \times (0,T], \tag{2.46}$$

$$\frac{\delta}{4}\partial_t \theta_\Gamma = \Delta_\Gamma \theta_\Gamma - \frac{1}{2}\Delta_\Gamma \mu_\Gamma - \frac{\delta\,(c_1 u(t) + c_2)}{4}\theta_\Gamma - \frac{(c_1 u(t) + c_2)}{2}\varphi_\Gamma \text{on } \Gamma \times (0,T], \tag{2.47}$$

together with

$$\frac{d}{dt}\int_B u(t) = -\frac{c_1}{|B|}\left(\int_B u(t)\right)^2 + \left(c_1\frac{M-|\Gamma|}{|B|} - c_2\right)\int_B u(t) + c_2 M \quad \text{on } (0,T], \tag{2.48}$$

$$\frac{d}{dt}\int_\Gamma \varphi = 0 \quad \text{on } (0,T], \tag{2.49}$$

$$\frac{\delta}{4}\int_\Gamma \theta = M - \int_B u - \int_\Gamma \frac{1+\varphi}{2} \quad \text{on } (0,T], \tag{2.50}$$

$$\int_\Gamma \mu = \int_\Gamma \left(\varepsilon^{-1}W'(\varphi) + \frac{\theta}{2}\right) \quad \text{on } (0,T], \tag{2.51}$$

which is an equivalent formulation for the reduced problem (2.28)–(2.44).

Chapter 3

Mathematical background

This chapter primarily serves as a brief collection of well-known results we use throughout this thesis. With a focus on function spaces and interpolation theory as well as geometric measure theory we establish the necessary mathematical tools and fix some notation. As such, this chapter surely does not provide a comprehensive introduction to interpolation theory or geometric measure theory but focuses on definitions and concepts used in this thesis. Therefore we begin each section with a list of references which provide a more detailed introduction to the respective topic.

In addition to some facts on function spaces and geometric measure theory as discussed in Sections 3.1 and 3.2 below we need some basic concepts from differential geometry, mainly (geodesic) mean curvature, the exponential map and representations of the surface gradient and the Laplace-Beltrami operator in local coordinates. We refer the reader to the books by do Carmo [dC92] or Bär [Bär10] for an introduction to these concepts.

3.1 Function Spaces

This thesis is concerned with weak solutions to partial differential equations. As usual, these solutions belong to the Sobolev spaces $W^{k,p}(B)$ or $W^{k,p}(\Gamma)$ respectively. It will sometimes be useful to also consider function spaces in the larger picture offered by interpolation theory. This section provides an introductory overview of those function spaces and tools from interpolation theory that are relevant to this work. For a complete treatment we refer the reader to the books by Triebel [Tri78], Lunardi [Lun09], or Adams and Fournier [AF03].

Let Γ be any compact Riemannian manifold or any open subset of $\mathbb{R}^n, k \in \mathbb{N}_0$, and $1 \leq p \leq \infty$. We will adopt the following notations and conventions.

(i) $C^k(\Gamma, \mathbb{R}^n)$ denotes all k-times differentiable functions $f : \Gamma \to \mathbb{R}^n$ such that $\partial_x^\alpha f$ has for all $\alpha \in \mathbb{N}_0^n, |\alpha| \leq k$ a continuous extension on $\overline{\Gamma}$.

(ii) $C_c^k(\Gamma, \mathbb{R}^n) := \{f \in C^k(\Omega, \mathbb{R}^n) \mid \operatorname{supp} f \subset \Omega \text{ is compact.}\}$. Its dual, the space of distributions is denoted by $D'(\Omega)$.

(iii) $C_0(\Gamma, \mathbb{R}^n)$ denotes the closure of $C_c^\infty(\Omega, \mathbb{R}^n)$ with respect to the supremums norm.

(iv) $L^p(\Gamma, \mathbb{R}^n)$ and $W^{k,p}(\Gamma, \mathbb{R}^n)$ are the usual Lebesgue and Sobolev spaces.

(v) $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions, i.e.

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \,\middle|\, x^\beta D^\alpha f \in L^\infty(\mathbb{R}^n) \text{ for all } \alpha, \beta \geq 0 \right\}.$$

Its dual, the so called space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

We will shorten the notation and write $C^k(\Omega) := C^k(\Omega, \mathbb{R}), L^p(\Omega) := L^p(\Omega, \mathbb{R}), \ldots$ and so on. All spaces are endowed with their usual norms.

Interpolation spaces allow a finer description of properties of functions. Colloquially speaking, an interpolation space X is a Banach spaces that lies between two suitable Banach spaces X_0 and X_1. The precise mathematical formulation of this idea is given in the two following definitions.

Definition 3.1 (Admissible Spaces)**.** Let X_0, X_1 be two $\mathbb{K}$–Banach spaces. The pair (X_0, X_1) is said to be admissible if there exists some topological Hausdorff space Z such that $X_0, X_1 \hookrightarrow Z$.

The space $X_0 \cap X_1$ endowed with the norm

$$\|x\|_{X_0 \cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$$

and the space $X_0 + X_1$ together with

$$\|x\|_{X_0+X_1} := \inf_{\substack{x = x_0 + x_1, \\ x_0 \in X_0, x_1 \in X_1}} \|x\|_{X_0} + \|x\|_{X_1}.$$

are Banach spaces for any admissible pair (X_0, X_1).

Definition 3.2 (Interpolation spaces)**.** Let (X_0, X_1) and (Y_0, Y_1) be admissible pairs. We say that

1. a space X is a intermediate space with respect to (X_0, X_1) if

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1.$$

2. intermediate spaces X (with respect to (X_0, X_1)) and Y (with respect to (Y_0, Y_1)) are interpolation spaces if for all linear operators $T : X_0 + X_1 \to Y_0 + Y_1$ the property

$$T \in \mathcal{L}(X_i, Y_i) \text{ for } i = 0, 1$$

already implies that

$$T\big|_X \in \mathcal{L}(X, Y).$$

3. interpolation spaces X and Y with respect to (X_0, X_1) and (Y_0, Y_1) respectively are of exponent $\theta \in [0, 1]$ if there exists a constant $C > 0$

$$\|T\|_{\mathcal{L}(X,Y)} \leq \|T\|^{1-\theta}_{\mathcal{L}(X_0,Y_0)} \|T\|^{\theta}_{\mathcal{L}(X_1,Y_1)} \text{ for all } T \in \mathcal{L}(X_i, Y_i) \text{ for } i = 0, 1.$$

There are several methods to construct interpolation spaces to a given admissible couple (X_0, X_1), leading to either real or complex interpolation spaces. Real interpolation spaces can be obtained via the K- or the trace method. Complex interpolation spaces are the result of the fittingly named complex method. Both real and complex interpolation spaces are thoroughly treated in [Lun09], Chapter 1 and Chapter 2 respectively. We shall only give the definition of real interpolation spaces via the K-method, which is sufficient to introduce the special cases we are interested in. For more details, we refer the reader to the aforementioned books [Tri78, Lun09].

To begin with, we define for $t > 0$ and $x \in X_0 + X_1$ the K-function K by

$$K(t,x) := \inf_{\substack{x = x_0 + x_1,\\ x_0 \in X_0, x_1 \in X_1}} \|x\|_{X_0} + t\,\|x\|_{X_1}\,.$$

Moreover, we denote by $\|\cdot\|_{L^p((0,\infty),\frac{dt}{t})}$ the weighted L^p-norms with respect to the measure $\frac{dt}{t}$ on $\mathbb{R}_+$.

Definition 3.3 (K-method for real interpolation)**.** Let $\theta \in (0,1)$ and $1 \leq p \leq \infty$. The spaces

$$(X_0,X_1)_{\theta,p} := \left\{ x \in X_0 + X_1 \,\middle|\, \left\| t^{-\theta} K(t,x) \right\|_{L^p((0,\infty),\frac{dt}{t})} < \infty \right\}$$

are called real interpolation spaces. They are normed by

$$\|x\|_{\theta,p} := \left\| t^{-\theta} K(t,x) \right\|_{L^p((0,\infty),\frac{dt}{t})}\,.$$

Proposition 3.4. *The spaces $(X_0,X_1)_{\theta,p}$ are Banach spaces for all $\theta \in (0,1)$ and $1 \leq p \leq \infty$. Moreover, they are interpolation spaces of exponent θ in the sense of Definition 3.2.*

Proof. This follows from Proposition 1.5 and Theorem 1.6 in [Lun09]. ☐

As a first application, we consider the Sobolev-Slobodeckij spaces on $\Omega \subset \mathbb{R}^n$ and on compact manifolds Γ which allow us to state the important trace theorem.

Definition 3.5 (Sobolev-Slobodeckij spaces)**.** Let $\Omega \subset \mathbb{R}^n$ be an open domain. For $s > 0, k = \lfloor s \rfloor$ and $1 \leq p < \infty$, the Sobolev-Slobodeckij space $W^{s,p}(\Omega)$ is the space of all $f \in W^{k,p}(\Omega)$ such that

$$\left[D^k f \right]_{s,p} < \infty \text{ where } [f]_{s,p} := \left(\int_{\Omega\times\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{(s-k)p+n}}\, dxdy \right)^{1/p}.$$

The norm on $W^{s,p}(\Omega)$ is given by

$$\|\cdot\|_{W^{s,p}(\Omega)} = \|\cdot\|_{W^{k,p}(\Omega)} + \left[D^k \,\cdot\, \right]_{s,p}.$$

Now let Γ be a compact Riemannian manifold. We say that $f : \Gamma \to \mathbb{R}$ is an element of $W^{s,p}(\Gamma)$ if it is locally in $W^{s,p}(U)$ for every coordinate patch U, where we identify U with its image in $\mathbb{R}^n$ under a suitable chart.

Lemma 3.6 ([Lun09, Example 1.8])**.** For $s \in (0,1)$ and $1 \leq p < \infty$, we have

$$W^{s,p}(\Omega) = (L^p(\Omega), W^{1,p}(\Omega))_{s,p}$$

with equivalent norms.

Theorem 3.7 (Trace Theorem)**.** *Let $1 \leq p < \infty$ and $s > 1/p$. Moreover, let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. For any continuous function $f \in C(\overline{\Omega})$ we define the trace operator* tr *as*

$$\operatorname{tr}(f) := f\big|_{\partial\Omega}.$$

Then tr *can be extended to a bounded linear operator*

$$\operatorname{tr} : W^{s,p}(\Omega) \to W^{s-\frac{1}{p},p}(\partial\Omega).$$

Proof. The proof is to be found in [AF03, Theorem 7.39]. □

A closely related class of function spaces are the Bessel potential spaces $H^{s,p}(\Omega)$. To shorten the notation in the actual definition, let

$$\langle\xi\rangle := \left(1 + |\xi|^2\right)^{1/2}.$$

Definition 3.8 (Bessel potential spaces)**.** Let $s \geq 0$ and let $1 < p < \infty$. Then the Bessel potential space $H^{s,p}(\mathbb{R}^n)$ is defined by

$$H^{s,p}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \mathcal{F}^{-1}\left[\langle\xi\rangle^s \mathcal{F}(f)\right] \in L^p(\mathbb{R}^n) \right\}$$

where $\mathcal{F}$ denotes the Fourier transformation. The space is normed by

$$\|f\|_{H^{s,p}(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}\left[\langle\xi\rangle^s \mathcal{F}(f)\right]\right\|_{L^p(\mathbb{R}^n)}.$$

For a bounded domain Ω with C^∞ boundary, we set

$$H^{s,p}(\Omega) := \left\{ f \in D'(\Omega) \,\middle|\, \exists\, g \in H^{s,p}(\mathbb{R}^n) \text{ with } g\big|_\Omega = f \right\}.$$

For a compact Riemannian manifold Γ we define $H^{s,p}(\Gamma)$ to be the space of all functions $f : \Gamma \to \mathbb{R}$ such that f is locally in $H^{s,p}(U)$ for every coordinate patch U, where again we identify U with its image in $\mathbb{R}^n$ under a suitable chart.

Remark 3.9. The relation between the spaces $H^{s,p}(\Omega)$ and $W^{s,p}(\Omega)$ is as follows. For general $1 < p < \infty$ and $s \in (0,1)$, the spaces $W^{s,p}(\Omega)$ and $H^{s,p}(\Omega)$ do not need to coincide, see the discussion in [Lun09, Example 2.12]. However, they do coincide if either s is an integer or in the special case $p = 2$. The proofs can be found in [Tri78, Section 2.3.2 and Section 2.3.3]. For the corresponding result on a compact manifold, we refer to [Tri92, Section 7.4.2 and Section 7.4.5].

Remark 3.10. The remark above yields that for $s \in (0,1)$ the spaces $H^s(\Gamma)$ and $H^s(\Omega)$ are interpolation spaces of exponent s by Lemma 3.6. A direct consequence is the interpolation inequality

$$\|f\|_{H^s(\Omega)} \leq C\,\|f\|_{L^2(\Omega)}^{1-s}\,\|f\|_{H^1(\Omega)}^{s}. \tag{3.1}$$

We note that our argument here is quite a detour, since the spaces $H^s(\Gamma)$ can also be directly obtained from complex interpolation, see [Tri92, Section 7.4.5] and also [Tay11, Section 13.6]. However, our argument has the advantage that we can omit the discussion of complex interpolation.

A useful technical consequence from the interpolation inequality (3.1) is the following lemma.

Lemma 3.11. Let X, X_0 and X_1 be Banach spaces such that $X_0 \cap X_1 \neq \varnothing$ and for $s \in (0,1)$ and some $C > 0$ the inequality

$$\|x\|_X \leq C \|x\|_{X_0}^{1-s} \|x\|_{X_1}^{s} \text{ for all } x \in X_0 \cap \mathcal{X}_1$$

holds. Furthermore, let $0 < T \leq \infty$ and $1 \leq p, p_0, p_1 \leq \infty$ with

$$\frac{1}{p} = \frac{1-s}{p_0} + \frac{s}{p_1}.$$

Then

$$\|f\|_{L^p(0,T;X)} \leq C \|f\|_{L^{p_0}(0,T;X_0)}^{1-s} \|f\|_{L^{p_1}(0,T;X_1)}^{s} \tag{3.2}$$

for all $f \in L^{p_0}(0,T;X_0) \cap L^{p_1}(0,T;X_1)$. Moreover, the embedding

$$L^p(0,T;X) \hookrightarrow L^{p_0}(0,T;X_0) \cap L^{p_1}(0,T;X_1)$$

is continuous.

Proof. Hölder's inequality directly implies (3.2). Indeed we have $\frac{(1-s)p}{p_0} + \frac{sp}{p_1} = 1$ and thus

$$\begin{aligned}\int_0^T \|f(t)\|_X^p \, dt &\leq C \int_0^T \|f(t)\|_{X_0}^{(1-s)p} \|f(t)\|_{X_1}^{sp} \, dt \\ &\leq C \left(\int_0^T \|f(t)\|_{X_0}^{p_0} \, dt \right)^{\frac{1}{p_0}(1-s)p} \left(\int_0^T \|f(t)\|_{X_1}^{p_1} \, dt \right)^{\frac{1}{p_1}sp},\end{aligned}$$

which already is (3.2). Furthermore, together with Young's inequality for the exponents $\frac{1}{1-s}$ and $\frac{1}{s}$ we infer

$$\|f\|_{L^p(0,T;X)} \leq C \|f\|_{L^{p_0}(0,T;X_0)}^{1-s} \|f\|_{L^{p_1}(0,T;X_1)}^{s} \leq C \|f\|_{L^{p_0}(0,T;X_0)} + C \|f\|_{L^{p_1}(0,T;X_1)}$$

from (3.2). Thus the embedding is continuous. □

The spaces $H^s(\Gamma)$ are Hilbert spaces if s is an integer. According to the Riesz representation Theorem 3.22, they are thus isometric to their dual spaces. We will mostly work with dual spaces in the following special cases.

Definition 3.12. Let $\Omega \subset \mathbb{R}^n$ be a domain with C^1-boundary. We then set

$$H_0^1(\Omega) = \left\{ f \in H^1(\Omega) \mid \operatorname{tr}(f) = 0 \right\} \text{ and } H^{-1}(\Omega) = H_0^1(\Omega)'.$$

If Γ is any smooth compact Riemannian manifold, then $H^{-1}(\Gamma)$ denotes the dual space of $H^1(\Gamma)$. Moreover, for $m_0 \in \mathbb{R}$ the subscript (m_0) will denote those subsets of function spaces on which all functions have mean value m_0, i.e we define

$$H_{(m_0)}^s(\Gamma) := \left\{ f \in H^s(\Gamma) \,\middle|\, \frac{1}{|\Gamma|} \int_\Gamma f \, d\mathcal{H}^n = m_0 \right\}.$$

We set $H_{(m_0)}^{-1}(\Gamma) := H_{(m_0)}^1(\Gamma)'$.

We now consider a special case relevant to the discussion of the lipid raft model (2.2)–(2.7). Let B and $\Gamma = \partial B$ be as in Chapter 2. We denote by $(-\Delta_N)^{-1} : H^{-1}_{(0)}(B) \to H^1_{(0)}(B)$ the operator that maps any element $F \in H^{-1}(B)$ such that $\langle F, 1\rangle = 0$ onto the solution $u \in H^1_{(0)}(B)$ of the weak Laplace operator in B, i.e. $u = (-\Delta_N)^{-1}F$ solves

$$\int_B \nabla u \cdot \nabla \eta = \langle F, \eta\rangle_{H^{-1}_{(0)}(B), H^1_{(0)}(B)}$$

for all $\eta \in H^1_{(0)}(B)$.

Note that $f \in L^2(B)$ and $g \in L^2(\Gamma)$ can be associated with a functional $F \in H^{-1}(B)$ by setting

$$\langle F, \eta\rangle_{H^{-1}(B), H^1(B)} := \int_B f\eta \; dx + \int_\Gamma g \operatorname{tr} \eta \; d\mathcal{H}^2.$$

If $\int_B f \; dx + \int_\Gamma g \; d\mathcal{H}^2 = 0$, the function $u = (-\Delta_N)^{-1}F$ is the weak solution to the Neumann problem

$$\begin{aligned} -\Delta u &= f \text{ in } B \\ \nabla u \cdot \nu &= g \text{ on } \Gamma. \end{aligned}$$

We will endow the spaces $H^1_0(B)$ and $H^1_{(0)}(B)$ with the norm $\|f\|_{H^1_0(B)} = \|f\|_{H^1_{(0)}(B)} = \|\nabla f\|_{L^2(B)}$, which in both cases are equivalent to the norm given in Definition 3.8. The dual norm is given via the isometry from the Riesz representation theorem (see e.g. [Yos95]). One readily checks that the Riesz isometry between $H^{-1}_{(0)}(B)$ and $H^1_{(0)}(B)$ is exactly the Neumann Laplace operator discussed above while one needs to consider $-\Delta_N + \mathrm{Id}$ in the case of $H^{-1}(B)$ and $H^1(B)$. Hence we have

$$\|F\|_{H^{-1}(B)} = \left\|(-\Delta_N + \mathrm{Id})^{-1}F\right\|_{H^1(B)} \text{ for all } F \in H^{-1}(B) \tag{3.3}$$

and

$$\|F\|_{H^{-1}_{(0)}(B)} = \left\|\nabla(-\Delta_N)^{-1}F\right\|_{L^2(B)} \text{ for all } F \in H^{-1}_{(0)}(B). \tag{3.4}$$

The analogue is true for $H^{-1}_{(0)}(\Gamma)$.

One of the most interesting questions when studying the relation between function spaces is the question about continuous embeddings from one function space into another. In general, many of the well-known embedding theorems for Sobolev spaces remain true if one considers functions defined on a compact, smooth, Riemannian manifold. We do not repeat these results here but refer the reader to the treatments by Aubin [Aub98] and Hebey [Heb96].

In particular, the well-known Sobolev embeddings are true for functions on a compact, smooth, Riemannian manifold, see [Aub98, Chapter 2] and [Heb96, Section 3.3].

To close this section, we introduce the family of Besov spaces. They are a further generalization of the Sobolev-Slobodeckij spaces defined above.

Definition 3.13 (Besov space)**.** Let $s > 0$ and $1 \le p, q < \infty$. Let Ω be a C^∞-domain. Moreover, let $k, l \in \mathbb{N}$ such that $0 \le k < s$ and $l > s - k$. Define

$$\Delta_h f(x) = f(x+h) - f(x) \text{ and } \Delta_h^l = \Delta_h(\Delta_k^{l-1}) \text{ for } l \ge 2.$$

The space of Besov function $B^s_{p,q}(\Omega)$ is the subspace of functions f in $L^p(\Omega)$ such that

$$\|f\|_{B^s_{p,q}(\Omega)} := \|f\|_{L^p(\Omega)} + \sum_{|\alpha|\leq k} \left[\int_{\mathbb{R}^n} |h|^{-(s-k)q} \left\| \Delta^l_h D^\alpha f \right\|_{L^p(\Omega_{h,l})} \frac{dh}{|h|^n} \right]^{1/q} < \infty,$$

where $\Omega_{h,l} = \bigcap_{j=0}^{l} \{ x | x + jh \in \Omega \}$.

Again the space $B^s_{p,q}(\Gamma)$ for a compact manifold Γ contains all functions f which locally for every coordinate patch U are elements in $B^s_{p,q}(U)$, where we identify U with its image under a suitable chart as usual.

Remark 3.14. The reader might realize that our definition of Besov spaces differs from the more common definition via Fourier transformation and a partition of unity in the frequency domain, as it is given for example in the books of Triebel [Tri78, Tri92]. Our choice here is motivated by the application in the proof of Lemma 8.10. The norm on $B^s_{p,q}(\Omega)$ given for example in [Tri78, Definition 2.3.1 and Definition 4.2.1] is equivalent to our choice in 3.13 by [Tri78, Theorem 4.4.2]. This holds also in the case of Besov spaces on a compact manifold Γ. We refer the reader to [Tri78, Section 3.6.1] for more details.

The norm in our definition also allows the observation that for $s > 0, s \notin \mathbb{N}$ and $p = q$, the Besov space $B^s_{p,p}(\Gamma)$ coincides with $W^{s,p}(\Gamma)$. Moreover, the results in [Tri92, Section 7.4.2] together with the Paley-Littlewood theorem [Tri92, Theorem 7.4.5] imply $B^s_{p,p}(\Gamma) = H^s_p(\Gamma)$ for $1 < p < \infty$ and $s > 0$. Together with Remark 3.9, we deduce $B^s_{2,2}(\Gamma) = W^{s,2}(\Gamma)$ for all $s > 0$.

Remark 3.15. A useful property when dealing with polynomial non-linearities in partial differential equation is the fact that the Bessel potential space $H^2(\Gamma)$ is a Banach algebra under suitable prerequisites on Γ. Assume that Γ is a two dimensional, compact Riemannian manifold. By [BCD11, Corollary 2.86], $B^s_{p,q}(\Gamma) \cap L^\infty(\Gamma)$ is a Banach algebra for every $1 \leq p, q \leq \infty$ and $s > 0$. In particular, $L^\infty(\Gamma) \cap H^2(\Gamma)$ is a Banach algebra since $L^\infty(\Gamma) \cap H^2(\Gamma) = L^\infty(\Gamma) \cap B^2_{2,2}(\Gamma)$ by the foregoing remark. Because $\dim \Gamma = 2$, the Sobolev embedding theorem implies that $H^2(\Gamma) \hookrightarrow C(\Gamma)$, i.e. $\|f\|_{L^\infty(\Gamma)} \leq C \|f\|_{H^2(\Gamma)}$ for all $f \in H^2(\Gamma)$. Thus $L^\infty(\Gamma) \cap H^2(\Gamma) = H^2(\Gamma)$.

3.2 Geometric Measure Theory and Varifolds

This section gives a brief introduction to concepts from geometric measure theory. For a detailed presentations of geometric measure theory we refer the reader to the books of Simon [Sim83], Federer [Fed69], and the more accessible book by Morgan [Mor16] as well as the paper by Allard [All72]. The books by Evans and Gariepy [EG15] and Ambrosio, Fusco, and Pallara [AFP00] offer a detailed introduction to BV−functions as well as some introductory material on (geometric) measure theory. In most cases, these are also the references for the proofs which we omit in this section.

We first define the notion of a $(n-1)$−varifold on some n−dimensional manifold as a way to generalize $(n-1)$−dimensional submanifolds. We also discuss the first variation of a varifold as a way to define the mean curvature of a general manifold. These concepts will be used in Chapter 8 to find a weak formulation of the sharp-interface limit $\varepsilon \searrow 0$ in the equations (2.2)–(2.7).

We then discuss how this is related to the definition of a generalized mean curvature for the boundary of sets of finite perimeter introduced by Röger [Rög04].

3.2.1 Radon Measures, Weak Convergence and Compactness

Throughout this section, $\mathcal{X}$ denotes a locally compact and separable metric space. We remark that in particular every Riemannian manifold is a locally compact, separable metric space. Hence the special case of $\mathcal{X}$ being a Reimannian (sub-)manifold of $\mathbb{R}^n$ considered in Section 3.2.2 is readily covered by the discussions in this section. The symbol Σ will always denote a $\sigma-$algebra of $\mathcal{X}$. The Borel σ-algebra (i.e. the smallest $\sigma-$algebra that contains all open subsets of $\mathcal{X}$) will be denoted by $\mathcal{B}(\mathcal{X})$.

Definition 3.16. A function $\lambda : \Sigma \to [0, \infty]$ is called measure if for $\{A_k\}_{k\in\mathbb{N}} \subset \Sigma$ such that $A_k \bigcap A_l = \varnothing$ for all $k \neq l$ the equality

$$\lambda\left(\bigcup_{k\in\mathbb{N}} A_k\right) = \sum_{k=0}^{\infty} \lambda(A_k)$$

holds. The tuple $(\mathcal{X}, \Sigma, \lambda)$ is called measure space. Moreover,

1. the measure λ is called Borel measure if $\mathcal{B}(\mathcal{X}) \subset \Sigma$.
2. the measure λ is called inner regular if for all $A \in \mathcal{B}(\mathcal{X})$
$$\lambda(A) = \sup\{\lambda(K)\,|K \subset A \text{ compact}\}.$$
3. the measure λ is called a Radon measure if it is a inner regular Borel measure and finite on compact sets.

Definition 3.17. Let λ be a measure on $\mathcal{X}$ and let $A \in \Sigma$. By $\lambda \llcorner A$ we denote the measure λ restricted to A, defined by

$$(\lambda \llcorner A)(B) = \lambda(A \cap B) \text{ for all } B \text{ such that } A \cap B \in \Sigma.$$

Definition 3.18. Let Σ be a $\sigma-$algebra of $\mathcal{X}$. A function $\lambda : \Sigma \to (-\infty, +\infty)$ is called a signed measure if for $\{A_k\}_{k\in\mathbb{N}} \subset \Sigma$ such that $A_k \bigcap A_l = \varnothing$ for all $k \neq l$ the equality

$$\lambda\left(\bigcup_{k\in\mathbb{N}} A_k\right) = \sum_{k=0}^{\infty} \lambda(A_k)$$

holds. We denote the space of all signed Radon measures on $\mathcal{X}$ by $\mathcal{M}(\mathcal{X})$. For any signed measure λ we define the variation measure $|\lambda| : \Sigma \to [0, \infty]$ by

$$|\lambda|\,(A) := \sup\left\{\sum_{k\in\mathbb{N}} |\lambda(A_k)| \,\middle|\, \bigcup_{\kappa\in\mathbb{N}} A_k \subset A, A_k \bigcap A_l = \varnothing \text{ for all } k \neq l\right\}.$$

Definition 3.19 (Absolute Continuity)**.** Let ν be a measure on $(\mathcal{X}, \Sigma)$ and let λ be a signed measure on $(\mathcal{X}, \Sigma)$. The measure λ is called absolutely continuous with respect to ν if for every $A \in \Sigma$ the property $\nu(A) = 0$ directly implies that $|\lambda|\,(A) = 0$. We write $\lambda \ll \nu$.

Definition 3.20 (Support of a Measure)**.** Let λ be a measure on $\mathcal{X}$. The support of λ, denoted by $\operatorname{supp}\lambda$ is defined as

$$\operatorname{supp}\lambda := \{x \in \mathcal{X} |\, \lambda(U) > 0 \text{ for all neighbourhoods } U \text{ of } x\}.$$

If $\lambda \in \mathcal{M}(\mathcal{X})$, we set $\operatorname{supp}\lambda := \operatorname{supp}|\lambda|$.

Definition 3.21 (Mutual Singularity)**.** Let ν, λ be measures on $(\mathcal{X}, \Sigma)$. The measures ν and λ are called mutual singular, denoted by $\nu \perp \lambda$, if there exists $A \in \Sigma$ such that $\nu(A) = 0$ and $\lambda(\mathcal{X} \setminus A) = 0$.

In the case that ν and λ are signed measures, we define $\nu \perp \lambda :\Leftrightarrow |\nu| \perp |\lambda|$.

The space of $\mathcal{M}(\mathcal{X})$ all signed Radon Measures can also be characterized as a dual space.

Theorem 3.22 (Riesz Representation Theorem)**.** *Let $\Phi : C_0(\mathcal{X}) \to \mathbb{R}$ be linear and continuous. Then there exists a unique signed Radon measure λ on $\mathcal{X}$ such that*

$$\Phi(f) = \int_{\mathcal{X}} f \; d\lambda \text{ for all } f \in C_0(\mathcal{X})$$

Moreover, $|\lambda|(\mathcal{X}) = \|\Phi\|$.

Proof. The proof can be found in [AFP00, Theorem 1.54] □

The duality between $C_0(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$ established in the foregoing theorem allows us to introduce the notation of weak* convergence of measures as the weak* convergence in $\mathcal{M}(\mathcal{X})$ seen as the dual space to $C_0(\mathcal{X})$.

Definition 3.23 (Weak* Convergence in the Sense of Measures)**.** Let $\lambda \in \mathcal{M}(\mathcal{X})$ and let $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathcal{M}(\mathcal{X})$. We say that λ_k converges locally to λ in the sense of measures or that λ_k converges locally weakly* to λ if and only if

$$\lim_{k\to\infty} \int_{\mathcal{X}} f \; d\lambda_k = \int_{\mathcal{X}} f \; d\lambda$$

for all $f \in C_c(\mathcal{X})$.

If λ and all $\lambda_k, k \in \mathbb{N}$ are finite, we say that λ_k converges to λ in the sense of measures or that λ_k converges weakly* to λ if and only if

$$\lim_{k\to\infty} \int_{\mathcal{X}} f \; d\lambda_k = \int_{\mathcal{X}} f \; d\lambda$$

for all $f \in C_0(\mathcal{X})$.

The advantage of measure theory in the context of singular limits lies in the following observation. Bounded sets in any reflexive space (e.g. for $1 < p < \infty$ and a domain $\Omega \subset \mathbb{R}^n$ the $L^p(\Omega)$–spaces or the Sobolev spaces $W^{1,p}(\Omega)$) are weakly compact, allowing us to deduce that every bounded sequence in these spaces must admit a weakly converging subsequence. Unfortunately, this is wrong if we consider a sequence which is only bounded in a non reflexive space, for example $L^1(\Omega)$. It turns out that if we identify a bounded sequence $\{f_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ with the (bounded) sequence of measures $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ given by

$$\lambda_k(A) = \int_A f_k(x) \; dx \text{ for all } A \in \mathcal{B}(\Omega),$$

we obtain a subsequence λ_{k_j} which converges with respect to the weak*–topology on $\mathcal{M}(\Omega)$ by the following theorem.

Theorem 3.24 (Weak* Compactness for Radon Measures)**.** *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence of finite Radon measures on $\mathcal{X}$ such that*

$$\sup_{k\in\mathbb{N}} |\lambda_k|(\mathcal{X}) < \infty.$$

Then there exists a weakly converging subsequence. Moreover, the map $\lambda \mapsto |\lambda|(\mathcal{X})$ is lower semicontinuous with respect to the weak* convergence.*

Proof. We refer the reader to [AFP00, Theorem 1.59]. □

Theorem 3.25 (Radon-Nikodym)**.** *Let ν be a measure on $(\mathcal{X},\Sigma)$ and let λ be a signed measure on $(\mathcal{X},\Sigma)$. Moreover, assume that there exists an increasing sequence of sets $\{A_k\}_{k\in\mathcal{X}}$ such that $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$ and $\mathcal{X} = \bigcup_{k\in\mathbb{N}} A_k$.*

Then there is a unique decomposition of the measure λ into the sum of two measures λ^a and λ^s (i.e. $\lambda = \lambda^a + \lambda^s$) which fulfil $\lambda^a \ll \nu$ and $\lambda^s \perp \nu$.

Furthermore, there exists a unique function $f \in L^1(\mathcal{X},\nu)$ such that $\lambda^a = f\nu$.

Proof. The proof is given in [AFP00, Theorem 1.28]. □

Remark 3.26. 1. The decomposition $\lambda = \lambda^a + \lambda^s$ is also called Lebesque decomposition, with λ^a being called the absolute continuous part and the singular part λ^s.

2. The function f such that $\lambda^a = f\nu$ is called density or Radon-Nikodym derivative of λ with respect to ν. It is often also denoted by $\frac{d\lambda}{d\nu}$.

3. In the case of Radon measures on $\mathbb{R}^n$, [EG15, Section 1.6] contains a more detailed discussion of derivatives of measures, motivating the name Radon-Nikodym derivative for f.

3.2.2 Rectifiable Sets and Varifolds

The aim of this section is to introduce the notion of a general varifold as a measure theoretic generalization of a submanifold. In the sharp-interface limit which is our main application, the object in question will be a generalization of a curve on a two dimensional submanifold of $\mathbb{R}^3$. Therefore for our purpose it is sufficient to assume from now on that $\mathcal{X}$ is a l–dimensional submanifold of $\mathbb{R}^n$. Of course, this requires $l \le n$.

We start this section with the more accessible special case of rectifiable varifolds, which consist of a pair (M,ω) of a rectifiable set M and a weight function ω.

Associated with the set M is its k–dimensional Hausdorff measure. For any given set $A \subset \mathcal{X}$, let the diameter $\operatorname{diam}(A)$ of A be given as $\operatorname{diam}(A) := \sup\{d(x,y) | x,y \in A\}$. We set $\operatorname{diam}(\varnothing) = 0$. Moreover, let $\mathcal{I}$ be a finite or countable index set. For $0 < \delta \le \infty$ and $k \in \mathbb{N}_0$ we introduce

$$\mathcal{H}^k_\delta(A) = \frac{|B_k|}{2^k} \inf\left\{ \sum_{i\in\mathcal{I}} [\operatorname{diam}(A_i)]^k \,\middle|\, \operatorname{diam}(A_i) < \delta, A \subset \bigcup_{i\in\mathcal{I}} A_i \right\},$$

where $|B_k|$ denotes the k–dimensional volume of the unit ball in $\mathcal{X}$.

Definition 3.27 (Hausdorff Measure)**.** For $k \in \mathbb{N}$ we define the k–dimensional Hausdorff measure of a set $A \in \mathcal{B}(\mathcal{X})$ as

$$\mathcal{H}^k(A) = \lim_{\delta\searrow 0} \mathcal{H}^k_\delta(A).$$

We refer the reader to [AFP00, Proposition 2.49] and Remark 2.48 therein for a proof that the Hausdorff measure defined above is indeed an outer measure on $\mathcal{X}$.

Remark 3.28. 1. It is also possible to define the Hausdorff measure for all $k \in [0,\infty)$ if one replaces $|B_k|$ with $\frac{\pi^{k/2}}{\Gamma(1+k/2)}$, where $\Gamma(s)$ is the Euler Gamma function. However, we will only use the Hausdorff measure for integer valued k.

2. If the set M is a smoothly embedded k−dimensional submanifold of R^n, then the Hausdorff measure $\mathcal{H}(\mathcal{M})$ coincides with the classical definition of the area of M as a submanifold (see [Mor16, Theorem 3.7]).

Definition 3.29 (Rectifiable Set)**.** A $\mathcal{H}^k$−measurable set $M \subset \mathcal{X}$ is called countably k−rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathcal{X}$ such that

$$M \subset \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k)$$

and if

$$\mathcal{H}^k\left(M \setminus \bigcup_{i=0}^{\infty} f_i\left(\mathbb{R}^k\right)\right) = 0.$$

In the following, M always denotes a k−rectifiable set and $\omega : M \to \mathbb{R}$ is a positive and locally $\mathcal{H}^k$−integrable functions on M. We say that (M,ω) and $(\widetilde{M},\widetilde{\omega})$ are equivalent rectifiable pairs if

$$\mathcal{H}\left(\left(M \setminus \widetilde{M}\right) \cup \left(\widetilde{M} \setminus M\right)\right) = 0 \qquad \text{and} \qquad \omega(x) = \widetilde{\omega}(x) \text{ for } \mathcal{H}^k - \text{almost all } x \in \mathcal{M}.$$

Definition 3.30 (Rectifiable Varifold)**.** Let (M,ω) be an equivalence class of k−rectifiable pairs. The associated rectifiable k−varifold on $\mathcal{X}$ is the Radon measure V defined by

$$V = \omega\mathcal{H}^k \llcorner M.$$

Definition 3.31 (Integral Varifold)**.** Let V be a rectifiable k−varifold with weight function ω. If ω is integer valued $\mathcal{H}^k$−almost everywhere, we say that V is a integral varifold.

By Whitney's Extension Theorem it is possible to show that a set $M \subset \mathbb{R}^n$ is k−rectifiable if and only if up to a Hausdorff null set M can be covered by countably many k−dimensional embedded C^1−submanifolds (see [Sim83, Lemma 11.1]). Thus M should Hausdorff almost everywhere carry a differentiable structure. This observation motivates the following definition of the approximate tangent spaces to M.

Definition 3.32 (Approximate Tangent Space of a Rectifiable Set)**.** Let $M \subset \mathcal{X}$ be a $\mathcal{H}^k$−measurable set and let $\omega : M \to \mathbb{R}$ be a positive and locally $\mathcal{H}^k$−integrable function. A k−dimensional subspace S of $T_p\mathcal{X}$ is called the approximate tangent space $T_p^{\text{app}}M$ for M at $p \in M$ with respect to ω, if

$$\lim_{\rho \searrow 0} \rho^{-k} \int_M \varphi(\rho^{-1}(q-p))\omega(q)\ d\mathcal{H}^k(q) = \omega(p) \int_S \varphi(q)\ d\mathcal{H}^k(q)$$

for all $\varphi \in C_0(\mathbb{R}^n)$.

Analogously, we define the approximate tangent space of a Radon measure.

Definition 3.33 (Approximate Tangent Space of a Radon Measure)**.** Let λ be a Radon measure on $\mathbb{R}^n$. For $\rho > 0$ we introduce the rescaled measures

$$\lambda_{x,\rho}(A) = \rho^{-k}\lambda(x + \rho A) \text{ for } A \subseteq \mathbb{R}^n \text{ open.}$$

Suppose that for λ almost all $x \in \mathbb{R}^n$ there is $\omega(x) \in (0,\infty)$ and a k−dimensional subspace $S \subset \mathbb{R}^n$ such that

$$\lim_{\rho \searrow 0} \int \varphi(y)\, d\lambda_{x,\rho}(y) = \omega(x) \int_S \varphi(y)\, d\mathcal{H}^k(y)$$

for all $\varphi \in C_0(\mathbb{R}^n)$. Then S is called the approximate tangent space $T_x^{\text{app}}\lambda$ with multiplicity $\omega(x)$.

If λ is a Radon measure on a smooth submanifold $\mathcal{X}$ of $\mathbb{R}^n$ instead of $\mathbb{R}^n$ we consider the measure $\lambda \llcorner \mathcal{X}$ on $\mathbb{R}^n$ and set $T_p^{\text{app}}\lambda := T_p^{\text{app}}\lambda \llcorner \mathcal{X}$ for $p \in \mathcal{X}$.

Remark 3.34. Any rectifiable pair (M,ω) in the Riemanian submanifold $\mathcal{X}$ is also a rectifiable pair in $\mathbb{R}^n$. Conversely, we can always consider the Radon measure $\lambda \llcorner \mathcal{X}$ on $\mathbb{R}^n$ for any Radon measure λ on $\mathcal{X}$. For this reason the two Definitions 3.32 and 3.33 transfer directly from the case where $M \subset \mathbb{R}^n$ to the case $M \subset \mathcal{X}$ and from $\lambda \in \mathcal{M}(\mathbb{R}^n)$ to $\lambda \in \mathcal{M}(\mathcal{X})$ respectively. We remark that this situation differs from the discussion of the generalized curvature of a varifold later, where we want to define the curvature of (M,ω) in $\mathcal{X}$ and not necessarily in $\mathbb{R}^n$.

Theorem 3.35. *Let λ be a Radon measure on $\mathbb{R}^n$. Moreover, assume that for λ almost all $x \in \mathbb{R}^n$ the approximate k−dimensional tangent space for λ in x exists with some multiplicity function ω fulfilling $\omega(x) \in (0,\infty)$. Let*

$$M := \{x \in \mathbb{R}^n \,| T_x^{app}\lambda \textit{ exists with multiplicity } \omega(x) \in (0,\infty)\}$$

and set $\omega(x) = 0$ for $x \in \mathbb{R}^n \setminus M$. Then M is countably k−rectifiable, ω is $\mathcal{H}^k$−measurable on $\mathbb{R}^n$, and $\lambda = \omega\mathcal{H}^k \llcorner M$.

Conversely, let $M \subset \mathbb{R}^n$ be countably k−rectifiable. Then there exists a positive, locally $\mathcal{H}^k$−integrable function ω on M such that $T_p^{app}M$ exists with respect to ω for $\mathcal{H}^k$−almost all $p \in M$.

Proof. The statement follows directly from Theorem 11.6 and Theorem 11.8 in [Sim83]. ☐

For a rectifiable varifold, it is thus sufficient to specify the corresponding weight function and the underlying rectifiable set M in order to determine its tangent space. As such, the following definition is justified.

Definition 3.36 (Tangent Space of a Rectifiable Varifold)**.** The tangent space of a rectifiable varifold V is the approximate tangent space to V as defined in Definition 3.33

For a general varifold, we abandon this definition of a tangent space and choose instead to include tangential information in the definition of the varifold.

To this end, let $\mathcal{X} \subset \mathbb{R}^n$ be a l−dimensional Riemanian manifold. By $\mathbb{S}^k(p)$ we denote for $k \leq l$ the Grassmanian of all k−dimensional subspaces of $T_p(\mathcal{X})$

$$\mathbb{S}^k(p) := \{S \,|S \text{ is a } k-\text{dimensional subspace of } T_p(\mathcal{X})\}\,.$$

To introduce a topology on $\mathbb{S}^k(p)$, we introduce

$$V^k T_p(\mathcal{X}) := \{(v_1,\ldots,v_k)|\, v_1,\ldots,v_l \in T_p(\mathcal{X}) \text{ and linearly independent}\}\,.$$

The topology on $\mathbb{S}^k(p)$ is then given as the quotient topology induced by the map

$$\pi : V^k T_p(\mathcal{X}) \to \mathbb{S}(p)$$

which sends a tuple of k linearly independent vectors in $T_p(\mathcal{X})$ onto the k–dimensional subspace they span.

Moreover, we define $G_k(\mathcal{X})$ as

$$G_k(\mathcal{X}) := \left\{ (p, S) \,|p \in \mathcal{X}, S \in \mathbb{S}^k(p) \right\}.$$

Since $G_k(\mathcal{X})$ is (at least locally for $U \subset \mathcal{X}$) diffeomorphic to $U \times \mathbb{S}^k(p)$, the topologies on $\mathcal{X}$ and $\mathbb{S}^k(p)$ induce a topology on $G_k(\mathcal{X})$, see for example [Lee97, Lemma 2.2].

Definition 3.37 (Varifold)**.** Let $\mathcal{X} \subset \mathbb{R}^n$ be a l–dimensional Riemanian manifold and let $G_k(\mathcal{X})$ be defined as above. A general k–varifold (varifold for short) on $\mathcal{X}$ is a Radon measure on $G_k(\mathcal{X})$.

Remark 3.38. It is useful to introduce the following notation.

1. The orthogonal projection of $T_p\mathcal{X}$ onto $S \in \mathbb{S}^k(p)$ will also be denoted by S.

2. For $S \in \mathbb{S}^k(p)$ we denote by δ_S the Dirac measure concentrated on S. That is, for a set $P \subset \mathbb{S}^k(p)$ we define
$$\delta_S(P) := \begin{cases} 1, & \text{if } S \in P \\ 0, & \text{else.} \end{cases}$$

3. Let $\mathcal{X}$ be an l–dimensional manifold. We identify $\mathbb{S}^{l-1}(p) \cong S_{l-1}(p) \mod \{e_1, -e_1\}$ where $S_{l-1}(p)$ is the $(l-1)$–sphere in $T_p(\mathcal{X})$ and e_1 is the first unit vector. As such, we identify $\mathbb{S}^{l-1}$ with the set of all unit normal vectors to unoriented $(l-1)$ planes in $T_p\mathcal{X}$.

Definition 3.39 (Weight Measure of a Varifold)**.** Let V be an varifold on $\mathcal{X}$. The measure m_V on $\mathcal{X}$ defined as

$$m_V(A) := V(\{(p, S)| \, p \in A, S \in \mathbb{S}^k(p)\}) = \int_{G_k(A)} dV(p, S)$$

is called weight measure of V.

Remark 3.40. 1. To make the very general definition of a k–varifold on $\mathcal{X}$ more transparent and to understand in which sense varifolds generalize submanifolds, we study the following example. Let M be a smooth k–dimensional submanifold of $\mathcal{X}$ without boundary. Then we introduce a corresponding varifold V by setting

$$dV(p, S) = d\mathcal{H}^k \llcorner M(p) \delta_{T_pM}(S).$$

In this case, the weight measure m_V corresponds to the surface area measure on M.

2. The Gauss Theorem for non-tangential vectorfields on a smooth manifold M links the first variation of the surface area of M with its geodesic curvature κ_g in $\mathcal{X}$. Let $\Psi(t, x)$: $(-\rho, \rho) \times M \to \mathcal{X}$ be a smooth family of diffeomorphisms such that $\Psi(0, p) = p$ and $M_t := \Psi(t, M)$

are smooth submanifolds of $\mathcal{X}$. Moreover, let $\xi(p) := \partial_t\big|_{t=0}\Psi(t,p)$. Then the Gauss Theorem for non-tangential vectorfields yields

$$\frac{d}{dt}\Big|_{t=0}\mathcal{H}(M_t) = \int_M \operatorname{div}_M \xi \; d\mathcal{H}^k = -\int_M \xi \cdot \kappa_g \nu_M \; d\mathcal{H}^k,$$

where $\nu_M(p) \in T_p\mathcal{X}$ denotes the unit normal vector of M at p. Moreover, for the submanifold M we can express the surface divergence div_M as

$$\operatorname{div}_M \xi = \operatorname{div} \xi - \nu \cdot D\xi\nu,$$

where ν is the outer unit normal of M. This observation motivates the definition of the first variation of a varifold as well as the following definition of the mean curvature.

Definition 3.41 (First Variation of a Varifold)**.** Let V be a varifold on $\mathcal{X}$. The first variation $\delta V : C_c^1(\mathcal{X}, T\mathcal{X}) \to \mathbb{R}$ is given as

$$\delta V(\xi) = \int_{G_k(\mathcal{X})} D_{\mathcal{X}}\xi(p) : (\mathrm{Id} - S \otimes S) \; dV(p,S)$$

for all $\xi \in C_c^1(\mathcal{X}, T\mathcal{X})$. Here $D_{\mathcal{X}}\xi(p)$ denotes the differential of ξ on $\mathcal{X}$.

Definition 3.42 (Mean Curvature of a Varifold)**.** Let V be a varifold on $\mathcal{X}$. If there is a m_V−measurable vector valued function $\vec{\kappa}_g$ such that

$$-\delta V(\xi) = \int_{\mathcal{X}} \vec{\kappa}_g(p) \cdot \xi(p) \; dm_V(p)$$

for all $C_c^1(\mathcal{X}, T\mathcal{X})$, then we say that $\vec{\kappa}_g$ is the mean curvature vector of V in $\mathcal{X}$.

Remark 3.43. We return to the case of M being a smooth k−dimensional submanifold of $\mathcal{X}$. For V as in Remark 3.40(1) we calculate

$$\begin{aligned}\delta V(\xi) &= \int_{G_k(\mathcal{X})} D_{\mathcal{X}}\xi(p) : (\mathrm{Id} - S \otimes S) \; d\mathcal{H}^k \llcorner M(p)\delta_{T_pM}(S) \\ &= \int_M \operatorname{div}_M \xi(p) \; d\mathcal{H}^k(p).\end{aligned}$$

where we have identified that for $S = T_pM$ with the corresponding unit normal and used that in this case the definition of the surface divergence yields $D_{\mathcal{X}}\xi(p) : (\mathrm{Id} - S \otimes S) = \operatorname{div}_M \xi(p)$. Using Remark 3.40(2) we conclude

$$\delta V(\xi) = -\int_M \xi \cdot \kappa_g \nu_M \; d\mathcal{H}^k,$$

which shows that the generalized mean curvature vector of V in $\mathcal{X}$ coincides with the classical geodesic mean curvature vector of M in $\mathcal{X}$.

3.2.3 Functions of Bounded Variation

Functions of bounded variation arise in many problems in variational calculus. This class of functions has become an important tool to study problems involving hypersurfaces which might develop discontinuities. In the context of geometric evolution equations involving curvature, it is therefore a natural question how to define a generalized curvature for the boundary of sets which are described by a function of bounded variation. Röger [Rög04] gave an answer to this question which we quickly recall here, together with the definition of functions of bounded variation.

We make again the assumption that $\mathcal{X}$ is an l–dimensional submanifold of $\mathbb{R}^n$. Note that we require $\mathcal{X} = \mathbb{R}^n$ in the last definition of this section.

Definition 3.44 (BV-functions)**.** Let u be a function in $L^1(\mathcal{X})$. The distributional gradient $\nabla_{\mathcal{X}} u$ of u is a linear bounded functional on $C_c^1(\mathcal{X}, T\mathcal{X})$ and is defined by

$$\nabla_{\mathcal{X}} u(\xi) := -\int_\Omega u \operatorname{div}_{\mathcal{X}} \xi.$$

If $\nabla_{\mathcal{X}} u$ can be extended as a bounded linear functional over $C_0(\mathcal{X}, T\mathcal{X})$, we say that u is a function of bounded variation, denoted by $u \in BV(\mathcal{X})$.

Remark 3.45. 1. We denote by $|\nabla_{\mathcal{X}} u|$ the Radon measure defined as

$$|\nabla_{\mathcal{X}} u|(A) := \sup_{\xi \in C_0(A),\ |\xi| \le 1} \int_A u \operatorname{div}_{\mathcal{X}} \xi \text{ for all open } A \subseteq \mathcal{X}.$$

2. The space $BV(\mathcal{X})$ equipped with the norm

$$\|u\|_{BV(\mathcal{X})} := \|u\|_{L^1(\mathcal{X})} + \sup_{\xi \in C_0(\mathcal{X}),\ |\xi| \le 1} \int_{\mathcal{X}} u \operatorname{div}_{\mathcal{X}} \xi$$

is a Banach space, see [AFP00, Chapter 3].

Definition 3.46 (Caccioppoli set)**.** A set $A \subseteq \mathcal{X}$ is said to be a Caccioppoli set or a set of finite perimeter if and only if its characteristic function χ_A is in $BV(\mathcal{X})$. If χ_A is in $BV(U)$ for an open subset $U \subset \mathcal{X}$, we say that A has locally finite perimeter.

Definition 3.47 (Reduced Boundary)**.** Let $A \subseteq \mathcal{X}$ be a set of locally finite perimeter. The reduced boundary $\partial^* A$ is the set of all $p \in \operatorname{supp} |\nabla_{\mathcal{X}} \chi_A|$ such that the limit

$$\nu_A(p) := \lim_{\rho \searrow 0} \frac{\nabla_{\mathcal{X}} \chi_A(B_\rho(p))}{|\nabla_{\mathcal{X}} \chi_A|(B_\rho(p))}$$

exists and satisfies $|\nu_A(p)| = 1$

Remark 3.48 (De Giorgi's Structure Theorem)**.** The Structure Theorem for sets of finite perimeter assures that the reduced boundary $\partial^* A$ of a set A of finite perimeter is a rectifiable varifold, see [AFP00, Theorem 3.59] or [EG15, Section 5.7.3, Theorem 2].

Based on De Giorgi's Structure Theorem it is tempting to define the generalized mean curvature of $\partial^* A$ as the curvature of the corresponding varifold. However, one has to proceed cautiously, as the following observation shows. Given a convergent sequence $\{\phi_h\}_{h>0} \in BV(\Gamma)$, we consider the reduced boundary $\partial^*\{\phi = 1\}$ for the limit function ϕ as $h \to 0$. This reduced boundary can not necessarily be described by the limit of the surface measures $|\nabla_\Gamma \phi_h| \to V$ since cancellations may occur if for example $\{\phi_h = 1\}$ is not connected for some $h > 0$. As a result, the Radon measure given as the limit of the surface measures might contain a hidden boundary on which it has double multiplicity. Hence the case $V \neq |\nabla_\Gamma \phi|$ is possible. This shows that a priori there might be two (or more) different varifolds which can be used to define the mean curvature of $\partial^*\{\phi = 1\}$. Using the concept of integral varifolds, Röger [Rög04] introduced a notion for the generalized mean curvature vector of the set $\partial^*\{\phi = 1\}$ if we are in the Euclidean space, that is if $\{\phi = 1\} \subset \mathbb{R}^n$. The crucial observation is the following proposition.

Proposition 3.49 ([Rög04])**.** *Let $U \subset \mathbb{R}^n$ be open, $E \subset U$, and $\chi_E \in BV(U)$. Assume that there are two integral $(n-1)$-varifolds μ_1, μ_2 on U such that for $i = 1, 2$ the following hold:*

$$\partial^* E \subset \operatorname{supp}(\mu_i), \tag{3.5}$$

$$\mu_i \text{ has locally bounded first variation with mean curvature vector } \vec{H}_{\mu_i}, \tag{3.6}$$

$$\vec{H}_{\mu_i} \in L^2_{loc}(\mu_i), s > n-1, s > 2. \tag{3.7}$$

Then

$$\vec{H}_{\mu_1}|_{\partial^* E} = \vec{H}_{\mu_2}|_{\partial^* E}$$

is satisfied $\mathcal{H}^{n-1}$-almost everywhere on $\partial^* E$.

Based on this proposition, the following definition is justified.

Definition 3.50 ([Rög04])**.** Let $E \subset U$ and $\chi_E \in BV(U)$, and assume that there exists an integral $(n-1)$-varifold μ on U satisfying (3.5)–(3.7). Then we call

$$\vec{H} := \vec{H}_\mu|_{\partial^* E}$$

the generalized mean curvature vector of $\partial^* E$ and define a scalar mean curvature by

$$\kappa := H \cdot \frac{\nabla \chi_E}{|\nabla \chi_E|} \text{ on } \partial^* E.$$

3.2.4 Schätzles Convergence Result for Varifolds with Mean Curvature Given by an Ambient Sobolev Function

We now introduce a convergence result for the mean curvature of a sequence of varifolds which was proved by Schätzle in [Sch01]. For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open and bounded. We assume $\{E_j\}_{j \in \mathbb{N}}$ to be a sequence of subsets of Ω such that $\chi_{E_j} \in BV(\Omega)$ for all $j \in \mathbb{N}$. De Giorgi's structure theorem [AFP00, Theorem 3.59] for BV–functions implies that

$$V_j := \left|\nabla \chi_{E_j}\right|$$

is a sequence of integral $(n-1)$–varifolds and we assume that they fulfil for all $j \in \mathbb{N}$

$$\int_\Omega d\left|\nabla \chi_{E_j}\right| \leq C_1$$

for some $C_1 > 0$. Assume now that there exists a sequence $\{\mu_j\}_{j \in \mathbb{N}} \subset W^{1,p}(\Omega)$, $\frac{n}{2} < p < n$ such that

$$\|\mu_j\|_{W^{1,p}(\Omega)} \leq C_1 \text{ for all } j \in \mathbb{N},$$

i.e. we assume that $\int_\Omega \left|\nabla \chi_{E_j}\right|$ and $\|\mu_j\|_{W^{1,p}(\Omega)}$ admit a common upper bound. Furthermore, we assume that the mean curvature of the varifolds V_j coincides with the trace of the functions μ_j, meaning that

$$\vec{H}_{V_j} = \mu_j \nu_j \text{ on } \partial^* E_j,$$

where

$$\nu_j := \frac{\nabla \chi_{E_j}}{\left|\nabla \chi_{E_j}\right|} \text{ on } \partial^* E_j.$$

The equality above should be understood in the weak sense

$$\int_\Omega \nabla\xi - \frac{\nabla \chi_{E_j}}{\left|\nabla \chi_{E_j}\right|} D\xi \frac{\nabla \chi_{E_j}}{\left|\nabla \chi_{E_j}\right|} \, d\left|\nabla \chi_{E_j}\right| = \int_\Omega \chi_{E_j} \operatorname{div}(\mu_j \xi)$$

for all $\xi \in C^1_c(\Omega; \mathbb{R}^n)$.

Finally, we assume that there is a function $\mu \in W^{1,p}(\Omega)$, a set $E \subset \Omega$ and a Radon measure V on $G_{n-1}(\mathbb{R}^n)$ such that

$$\begin{aligned} \mu_j &\rightharpoonup \mu \text{ in } W^{1,p}(\Omega) \\ \chi_{E_j} &\to \chi_E \text{ in } L^1(\Omega) \\ V_j &\to V \text{ as varifolds.} \end{aligned}$$

Theorem 3.51 ([Sch01, Theorem 1.1]). *Under the above assumptions, V is an integral $(n-1)$–varifold in $\Omega \subseteq \mathbb{R}^n$ with locally bounded first variation and mean curvature vector*

$$\vec{H}_V \in L^s_{loc}(m_V) \textit{ for } s = \frac{-np}{p - n(n+1)}.$$

The set E is a set of finite perimeter in Ω and its reduced boundary is contained in the support of V that is

$$\partial^* E \subseteq \operatorname{supp} V.$$

The mean curvature vector satisfies

$$\vec{H}_V = \mu \nu_E \quad m_V - \textit{almost everywhere on } \operatorname{supp} V,$$

where $\nu_E = \frac{\nabla \chi_E}{|\nabla \chi_E|}$ denotes the generalized normal of $\partial^* E$, which is set to be equal to 0 outside of $\partial^* E$.

Chapter **4**

Existence of Solutions to the Full and the Reduced Model

4.1 Existence of Solutions to the Full Model (2.2)–(2.7)

The existence proof given here is based on the Galerkin method. The idea is to approximate the equations (2.2)–(2.7) by ordinary differential equations. We can thus obtain a sequence of approximative solutions to (2.2)–(2.7). The strategy is then to show that this sequence converges to a solution for the original problem.

We begin with the introduction of the function space $\mathcal{W}$, to which the weak solutions to problem (2.2)–(2.7) obtained from the Galerkin method will belong. Define

$$\mathcal{W} := \mathcal{W}_B \times \mathcal{W}_\Gamma^1 \times \mathcal{W}_\Gamma^2 \times \mathcal{W}_\Gamma^3 \times \mathcal{W}_\Gamma^4,$$

where

$$\begin{aligned}
\mathcal{W}_B &:= L^2(0,T;H^1(B)) \cap H^1(0,T;H^{-1}(B)),\\
\mathcal{W}_\Gamma^1 &:= L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma)),\\
\mathcal{W}_\Gamma^2 &:= L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma)),\\
\mathcal{W}_\Gamma^3 &:= L^2(0,T;H^1(\Gamma)), \text{ and}\\
\mathcal{W}_\Gamma^4 &:= L^2(0,T;H^1(\Gamma)).
\end{aligned}$$

The following lemma will be helpful while discussing this limit process for the exchange term q under suitable assumptions on q and the involved sequences.

Lemma 4.1. Let $T > 0$ and $\{u_k\}_{k\in\mathbb{N}} \in L^2(0,T;H^1(B))$ and $\{v_k\}_{k\in\mathbb{N}} \in L^2(0,T;H^1(\Gamma))$ be sequences such that

$$\|u_k\|_{L^2(0,T,H^1(B))} \le C(T) \qquad \text{and} \qquad \|v_k\|_{L^2(0,T,H^1(\Gamma))} \le C(T)$$

for some constant $C(T) > 0$. Additionally, assume that

$$\|\partial_t u_k\|_{L^2(0,T,(H^1(B))')} \le C(T) \qquad \text{and} \qquad \|\partial_t v_k\|_{L^2(0,T,H^{-1}(\Gamma))} \le C(T).$$

Moreover, let $q : \mathbb{R}^2 \to \mathbb{R}$ be continuous and assume that

$$\|q(\mathrm{tr}(u_k), v_k)\|_{L^2(0,T;L^2(\Gamma))} \le C(T).$$

Then we have subsequences of $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ (again denoted by $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ respectively) such that $u_k \to u \in L^2(0,T;H^s(B))$ for $0 < s < 1$ as well as $v_k \to v \in L^2(0,T;L^2(\Gamma))$. Furthermore,

$$q(\mathrm{tr}(u_k), v_k) \rightharpoonup q(\mathrm{tr}(u), v) \text{ in } L^2(0,T;L^2(\Gamma)).$$

Proof. The Aubin-Lions theorem [Sim87, Corollary 2] applied to $H^1(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow H^{-1}(\Gamma)$ allows us to deduce the relative compactness of $\{v_k\}$ in $L^2([0,T]\times\Gamma)$ and consequently (up to the extraction of a subseqeunce) the convergence of $v_k(x)$ pointwise almost everywhere in $\Gamma \times [0,T]$.

A similar pointwise convergence result holds for the sequence $\mathrm{tr}(u_k(x))$ on Γ. We can indeed again apply the Aubin-Lions theorem. In contrast to the easier case above, we have to account for the fact that we need some regularity of the limit function in order to make sense of the boundary values. To this end, we use that the embedding $H^1(B) \hookrightarrow H^s(B)$ is compact for all $1/2 < s < 1$, which allows us to work with $H^1(B) \hookrightarrow H^s(B) \hookrightarrow H^{-1}(B)$. After possibly extracting a subsequence, the Aubin-Lions theorem then yields the strong convergence $u_k \to u$ in $L^2(0,T;H^s(B))$ and by the continuity of the trace operator, we deduce $\mathrm{tr}(u_k) \to \mathrm{tr}(u)$ in $L^2([0,T]\times\Gamma)$. This directly implies the pointwise convergence $u_k(x) \to u(x)$ (again up to the extraction of a subsequence) almost everywhere in $[0,T]\times\Gamma$. The pointwise convergence results on v_k and u_k above are required in order to deduce the weak convergence of the nonlinearity $q(\mathrm{tr}(u_k), v_k)$ on Γ. Since by assumption $q(\mathrm{tr}(u_k), v_k)$ is bounded in the reflexive space $L^2(0,T;L^2(\Gamma))$, we deduce the existence of a function $\tilde{q} \in L^2(0,T;L^2(\Gamma))$ such that

$$q(\mathrm{tr}(u_k), v_k) \rightharpoonup \tilde{q} \text{ in } L^2(0,T;L^2(\Gamma)).$$

At the same time, $q(\mathrm{tr}(u_k), v_k)$ converges pointwise almost everywhere to $q(\mathrm{tr}(u), v)$ on $[0,T]\times\Gamma$ thanks to the continuity of q and the convergence results on u_k and v_k above. Since by [DiB02, Proposition 9.1c] pointwise and weak limit must coincide (if they both exist as in this case), we obtain the weak convergence

$$q(\mathrm{tr}(u_k), v_k) \rightharpoonup q(\mathrm{tr}(u), v) \text{ in } L^2(0,T;L^2(\Gamma)). \qquad \square$$

Theorem 4.2. *In accordance with Chapter 2 we choose the double-well potential $W(s) = (s^2-1)^2$ in the Ginzburg-Landau part of $\mathcal{F}$. Let $\varphi_0 \in H^1(\Gamma), v_0 \in H^1(\Gamma)$ and $u_0 \in L^2(B)$ be such that*

$$\mathcal{F}(v_0, \varphi_0) + \frac{1}{2}\int_B u_0^2 \le C.$$

Moreover, assume that the exchange term $q = q(u,v)$ is continuous and fulfils

$$|q(u,v)| \le C(1 + |u| + |v|).$$

Then there exist functions $(u, \varphi, v, \mu, \theta) \in \mathcal{W}$ which are a weak solution to problem (2.2)–(2.7)*,* *i.e. they fulfil for all $\xi \in L^2(0,T;H^1(B))$ and $\eta \in L^2(0,T;H^1(\Gamma))$ the equations*

$$\int_0^T \langle \partial_t u, \xi \rangle_{(H^1(B))', H^1(B)} = -\int_0^T \int_B \nabla u \cdot \nabla \xi - \int_0^T \int_\Gamma q(u,v)\xi, \tag{4.1}$$

$$\int_0^T \langle \partial_t \varphi, \eta \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} = -\int_0^T \int_\Gamma \nabla_\Gamma \mu \cdot \nabla_\Gamma \eta, \tag{4.2}$$

$$\int_0^T \int_\Gamma \mu\eta = -\int_0^T \int_\Gamma \left[\varepsilon \nabla_\Gamma \varphi \cdot \nabla_\Gamma \eta + \frac{1}{\varepsilon} W'(\varphi)\eta\right] - \frac{1}{\delta} \int_0^T \int_\Gamma (2v - 1 - \varphi)\,\eta, \tag{4.3}$$

$$\int_0^T \langle \partial_t v, \eta \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} = -\int_0^T \int_\Gamma \nabla_\Gamma \theta \cdot \nabla_\Gamma \eta + \int_0^T \int_\Gamma q(u,v)\eta, \tag{4.4}$$

$$\theta = \frac{2}{\delta}\,(2v - 1 - \varphi)\,. \tag{4.5}$$

The last equation holds in $L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma))$. The initial values are attained in $L^2(B)$ and $L^2(\Gamma)$ respectively. Moreover, (2.27) *from Corollary 2.6 holds.*

Proof. Let $\{\omega_i\}_{i\in\mathbb{N}}$ be the family of eigenfunctions of the Laplace-Beltrami-Operator Δ_Γ on the surface Γ and let $\{\lambda_i\}_{i\in\mathbb{N}}$ be the family of the corresponding eigenvalues. Analogously, we define $\{\kappa_i\}_{i\in\mathbb{N}}$ to be the family of eigenfunctions to the Laplace-Operator on B with (homogeneous) Neumann boundary conditions and denote by l_i the corresponding eigenvalues.

We now restrict ourselves to functions of the form

$$\begin{aligned}
u^N(t,x) &= \sum_{i=1}^N c^i_{u,N}(t)\kappa_i(x),\\
\varphi^N(t,x) &= \sum_{i=1}^N d^i_{\varphi,N}(t)\omega_i(x),\\
\mu^N(t,x) &= \sum_{i=1}^N d^i_{\mu,N}(t)\omega_i(x),\\
v^N(t,x) &= \sum_{i=1}^N d^i_{v,N}(t)\omega_i(x),
\end{aligned}$$

which are elements of the finite dimensional function spaces $V_\Gamma^N := \operatorname{span}\left(\{\omega_i\}_{i=1}^N\right)$ and $V_B^N := \operatorname{span}\left(\{\kappa_i\}_{i=1}^N\right)$ respectively. In accordance with (2.7) we set

$$\theta^N(t,x) = \frac{2}{\delta}\left(2d^1_{v,N}(t) - 1 - d^1_{\varphi,N}(t)\right)\omega_1 + \frac{2}{\delta}\sum_{i=2}^N \left(2d^i_{v,N}(t) - d^i_{\varphi,N}(t)\right)\omega_i.$$

The weak formulation of (2.2)–(2.7) for test functions $\omega \in \operatorname{span}\left(\{\omega_i\}_{i=1}^N\right)$ and $\kappa \in \operatorname{span}\left(\{\kappa_i\}_{i=1}^N\right)$ then reads

$$\int_B \partial_t u^N \kappa = -D\int_B \nabla u^N \cdot \nabla \kappa - \int_\Gamma q(u^N, v^N)\kappa, \tag{4.6}$$

$$\int_\Gamma \partial_t \varphi^N \omega = -\int_\Gamma \nabla_\Gamma \mu^N \cdot \nabla_\Gamma \omega, \tag{4.7}$$

$$\int_\Gamma \mu^N \omega = \int_\Gamma \left[\varepsilon \nabla_\Gamma \varphi^N \cdot \nabla_\Gamma \omega + \varepsilon^{-1} W'(\varphi^N)\omega - \delta^{-1}(2v^N - 1 - \varphi^N)\omega\right], \tag{4.8}$$

$$\int_\Gamma \partial_t v^N \omega = -\int_\Gamma \nabla_\Gamma \theta^N \cdot \nabla_\Gamma \omega + \int_\Gamma q(u^N, v^N)\omega. \tag{4.9}$$

Choosing $\kappa = \kappa_i$ and $\omega = \omega_i$ in (4.6)–(4.9) above yields the following system of the ordinary differential equations

$$\begin{aligned}
\partial_t c^i_{u,N}(t) &= l_i c^i_{u,N} - \int_\Gamma q\left(\sum_{j=1}^N c^j_{u,N}\kappa_j, \sum_{j=1}^N d^j_{v,N}\omega_j\right)\kappa_i,\\
\partial_t d^i_{\varphi,N} &= \lambda_i d^i_{\mu,N},\\
d^i_{\mu,N} &= \varepsilon\lambda_i d^i_{\varphi,n} + \frac{1}{\varepsilon}W'\left(\sum_{j=1}^N d^j_{\varphi,N}\omega_j\right)\omega_i - \frac{1}{\delta}\left(2d^i_{v,N} - d^i_{\varphi,N} - \delta_{1i}\right),\\
\partial_t d^i_{v,N} &= -\lambda_i \frac{2}{\delta}\left(d^i_{v,N} - d^i_{\varphi,N} - \delta_{1i}\right) + q\left(\sum_{j=1}^N c^j_{u,N}\kappa_j, \sum_{j=1}^N d^j_{v,N}\omega_j\right)\omega_i
\end{aligned}$$

for the coefficients $c^i_{u,N}, d^i_{\varphi,N}, d^i_{\mu,N}$ and $d^i_{v,N}$.

The system is complemented by initial conditions derived from the initial data u_0, φ_0, v_0. To this end, set the initial conditions for the above system to be $c^i_{u,N}(0) = \int_B u_0\kappa_i$, $d^i_{\varphi,N}(0) = \int_\Gamma \varphi_0\omega_i$ and so forth. Solutions of this system exist due to the theory of ODEs on some interval $(0, T_n), T_n > 0$. We simplify the notation and denote these solutions by $c^i_{u,N}, d^i_{\varphi,N}, d^i_{\mu,N}$ and $d^i_{v,N}$. Accordingly, we write

$$\begin{aligned}
u^N(t,x) &= \sum_{i=1}^N c^i_{u,N}(t)\kappa_i(x),\\
\varphi^N(t,x) &= \sum_{i=1}^N d^i_{\varphi,N}(t)\omega_i(x),\\
\mu^N(t,x) &= \sum_{i=1}^N d^i_{\mu,N}(t)\omega_i(x),\\
v^N(t,x) &= \sum_{i=1}^N d^i_{v,N}(t)\omega_i(x),\\
\theta^N(t,x) &= \frac{2}{\delta\sqrt{|\Gamma|}}\left(2d^1_{v,N}(t) - 1 - d^1_{\varphi,N}(t)\right) + \frac{2}{\delta}\sum_{i=2}^N\left(2d^i_{v,N}(t) - d^i_{\varphi,N}(t)\right)\omega_i.
\end{aligned}$$

We shall now derive estimates that prove that the solutions $c^i_{u,N}, d^i_{\varphi,N}, d^i_{\mu,N}$ and $d^i_{v,N}$ can be extended to the interval $(0,T)$ for every $N \in \mathbb{N}$ and that the sequences $\{u^N\}, \{\varphi^N\}$, $\{\mu^N\}$, and $\{v^N\}$ converge to suitable limit functions u, μ, φ and v.

Thus we need to derive uniform estimates which not only allow us to deduce the existence of limit functions but will also clarify in which sense the convergence should be understood. It remains then to show that the limit functions u, μ, φ and v solve the equations (2.2)–(2.6).

We begin by noting that $\kappa = u^N$ is an admissible test function in (4.6), since clearly $u^N \in \operatorname{span}\left(\{\kappa_i\}_{i=1}^N\right)$ by the definition of u^N. Choosing $\kappa = u^N$ in (4.6) yields

$$\frac{1}{2}\frac{d}{dt}\left[\int_B |u^N|^2\right] = \int_B u^N\partial_t u^N = -D\int_B |\nabla u^N|^2 - \int_\Gamma q(u^N, v^N)u^N$$

where we have used that the time dependent coefficients $c^i_{u,N}(t)$ are solutions to the ODE system above and therefore differentiable in t.

Analogously, one has that μ^N, θ^N and $-\partial_t\varphi^N$ are elements of span$\left(\{\omega_i\}_{i=1}^N\right)$ and therefore are admissible test functions in (4.7)–(4.9). Choosing $\omega = -\partial_t\varphi^N$ in (4.8), we obtain

$$\begin{aligned}\int_\Gamma -\partial_t\varphi^N\mu^N &= \int_\Gamma \left[-\varepsilon\nabla_\Gamma\varphi^N\cdot\nabla_\Gamma\left(\partial_t\varphi^N\right) - \varepsilon^{-1}W'(\varphi^N)\partial_t\varphi^N\right] + \frac{1}{2}\int_\Gamma \theta^N\partial_t\varphi^N \\ &= -\frac{d}{dt}\left[\int_\Gamma \frac{\varepsilon}{2}\left|\nabla_\Gamma\varphi^N\right|^2 + \frac{1}{\varepsilon}W(\varphi^N)\right] + \frac{1}{2}\int_\Gamma \theta^N\partial_t\varphi^N.\end{aligned}$$

Choosing $\omega = \mu^N$ in (4.8) leads to

$$\int_\Gamma \partial_t\varphi^N\mu^N = -\int_\Gamma \left|\nabla_\Gamma\mu^N\right|^2.$$

Finally, we use that $\partial_t v^N = \frac{\delta}{4}\partial_t\theta + \frac{1}{2}\partial_t\varphi^N$ to infer

$$\begin{aligned}\frac{\delta}{2}\frac{d}{dt}\left[\int_\Gamma \left|\theta^N\right|^2\right] + \frac{1}{2}\int_\Gamma \partial_t\varphi^N\theta^N &= \frac{\delta}{4}\int_\Gamma \theta^N\partial_t\theta^N + \frac{1}{2}\int_\Gamma \partial_t\varphi^N\theta^N \\ &= -\int_\Gamma \left|\nabla_\Gamma\theta^N\right|^2 + \int_\Gamma q(u^N, v^N)\theta^N\end{aligned}$$

from (4.9) with $\omega = \theta^N$.

We add these four equations to obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\int_B \left|u^N\right|^2 + \frac{\varepsilon}{2}\int_\Gamma \left|\nabla_\Gamma\varphi^N\right|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi^N) + \frac{\delta}{8}\int_\Gamma \left|\theta^N\right|^2\right] \\ &\qquad + D\int_B \left|\nabla u^N\right|^2 + \int_\Gamma \left|\nabla_\Gamma\mu^N\right|^2 + \int_\Gamma \left|\nabla_\Gamma\theta^N\right|^2 = \int_\Gamma q(u^N, v^N)(\theta^N - u^N). \qquad (4.10)\end{aligned}$$

In order to estimate the right-hand side, we use Hölder's and Young's inequality to estimate

$$\begin{aligned}\left|\int_\Gamma q(u^N, v^N)(\theta^N - u^N)\right| &\leq \frac{1}{2}\int_\Gamma \left|\theta^N - u^N\right|^2 + \frac{1}{2}\int_\Gamma \left|q(u^N, v^N)\right|^2 \\ &\leq \int_\Gamma \left|\theta^N\right|^2 + \int_\Gamma \left|u^N\right|^2 + C\int_\Gamma \left(1 + \left|u^N\right|^2 + \left|v^N\right|^2\right) \\ &\leq \int_\Gamma \left|\theta^N\right|^2 + C\int_\Gamma \left|u^N\right|^2 + C\left(1 + \int_\Gamma \left|v^N\right|^2\right). \qquad (4.11)\end{aligned}$$

Taking into account that $2v^N = \frac{\delta}{2}\theta^N + 1 + \varphi^N$ we derive

$$\left|v^N\right|^2 \leq C\left(\delta^2\left|\theta^N\right|^2 + \left|1 + \varphi^N\right|^2\right)$$

from Young's inequality. Since $\left|1 + \varphi^N\right|^2 \leq C(\varepsilon)\left(1 + \frac{1}{\varepsilon}W(\varphi^N)\right)$ we thus obtain

$$\int_\Gamma \left|v^N\right|^2 \leq C(\delta, \varepsilon)\left(1 + \frac{\delta}{8}\int_\Gamma \left|\theta^N\right|^2 + \frac{1}{2\varepsilon}\int_\Gamma W(\varphi^N)\right).$$

Therefore,

$$\int_\Gamma q(u^N, v^N)(\theta^N - u^N) \leq C(\delta, \varepsilon)\left[1 + \frac{1}{2}\int_\Gamma \left|u^N\right|^2 + \frac{\varepsilon}{2}\int_\Gamma \left|\nabla_\Gamma\varphi^N\right|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi^N) + \frac{\delta}{8}\int_\Gamma \left|\theta^N\right|^2\right] \qquad (4.12)$$

Combining (4.10) and (4.12) we arrive at

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\int_B |u^N|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma \varphi^N|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi^N) + \frac{\delta}{8}\int_\Gamma |\theta^N|^2\right] \\ + D\int_B |\nabla u^N|^2 + \int_\Gamma |\nabla_\Gamma \mu^N|^2 + \int_\Gamma |\nabla_\Gamma \theta^N|^2 \\ \le C(\delta)\left[1 + \frac{1}{2}\int_\Gamma |u^N|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma \varphi^N|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi^N) + \frac{\delta}{8}\int_\Gamma |\theta^N|^2\right],\end{aligned}$$

which allows us to employ Gronwall's inequality to deduce bounds on u^N, ϕ^N, μ^N and v^N provided we can control $\int_\Gamma |u^N|^2$. Using the trace theorem 3.7 and (3.1), we immediately find

$$\int_\Gamma |u^N|^2 \le C\left\|u^N\right\|^2_{H^{1/2}(B)} \le C(a)\left\|u^N\right\|^2_{L^2(B)} + \frac{1}{a}\left\|\nabla u^N\right\|^2_{L^2(B)}$$

for $a > 0$ arbitrary small. Choosing a small enough, we thus conclude

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\int_B |u^N|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma \varphi^N|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi^N) + \frac{\delta}{8}\int_\Gamma |\theta^N|^2\right] \\ + \frac{D}{2}\int_B |\nabla u^N|^2 + \int_\Gamma |\nabla_\Gamma \mu^N|^2 + \int_\Gamma |\nabla_\Gamma \theta^N|^2 \\ \le C(\delta)\left[1 + \frac{1}{2}\int_B |u^N|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma \varphi^N|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi^N) + \frac{\delta}{8}\int_\Gamma |\theta^N|^2\right].\end{aligned}$$

We are now in the position to apply Gronwall's inequality and after integrating the above equation in time from 0 to $T > 0$ we deduce

$$\begin{aligned}\sup_{0\le t\le T}\left\{\frac{1}{2}\int_B |u^N|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma \varphi^N|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi^N) + \frac{\delta}{8}\int_\Gamma |\theta^N|^2\right\} \\ + \frac{D}{2}\int_0^T\int_B |\nabla u^N|^2 + \int_0^T\int_\Gamma |\nabla_\Gamma \mu^N|^2 + \int_0^T\int_\Gamma |\nabla_\Gamma \theta^N|^2 \le C(T).\end{aligned} \tag{4.13}$$

Moreover, choosing $\omega = \omega_1 \equiv \text{const}$ in (4.8) yields

$$\int_\Gamma \mu^N = \frac{1}{\varepsilon}\int_\Gamma W'(\varphi^N) - \frac{1}{2}\int_\Gamma \theta^N.$$

Since $\varphi^N \in L^\infty(0,T;H^1(\Gamma))$ by (4.13), the Sobolev embedding theorem in dimension $\dim \Gamma = 2$ implies $\varphi^N \in L^\infty(0,T;L^p(\Gamma))$ for all $1 \le p < \infty$. As $W'(\varphi) = 4\left(\varphi^N\right)^3 - \varphi^N$ and $\left|\int_\Gamma \theta\right| \le C(\Gamma)\left\|\theta\right\|_{L^2(\Gamma)}$ we thus infer that

$$\sup_{0\le t\le T}\left|\int_\Gamma \mu^N(t)\right| \le C\left(\left\|\varphi\right\|_{L^\infty(0,T;H^1(\Gamma))} + \left\|\theta\right\|_{L^1\infty(0,T;L^2(\Gamma))}\right) \le C(T) \tag{4.14}$$

by (4.13). As a result, we obtain $\left\|\mu\right\|_{L^2(0,T;H^1(\Gamma))} \le C(T)$ from Poincaré's inequality, (4.13), and (4.14).

For any $\tau \in H^1(B)$, there exists $\tau_1 \in \operatorname{span}\{\kappa_i\}_{i\in\mathbb{N}}^N$ such that $\tau_2 := \tau - \tau_1$ is orthogonal to $\operatorname{span}\{\kappa_i\}_{i\in\mathbb{N}}^N$ in $L^2(B)$. Therefore $\langle \partial_t u^N, \tau\rangle = \int_B \partial_t u^N \tau_1$ and since τ_1 is an admissible test function in (4.6), we find

$$\begin{aligned}\left|\langle \partial_t u^N, \tau\rangle\right| = \left|\int_B \partial_t u^N \tau_1\right| \le D\left|\int_B \nabla u^N \cdot \nabla \tau_1\right| + \left|\int_\Gamma q(u^N, v^N)\tau_1\right| \\ \le D\left\|u^N\right\|_{H^1(B)}\left\|\tau_1\right\|_{H^1(B)} + \left\|q(u^N, v^N)\right\|_{L^2(\Gamma)}\left\|\tau_1\right\|_{L^2(\Gamma)}.\end{aligned}$$

Observe that the Trace Theorem 3.7 ensures $\|\tau_1\|_{L^2(\Gamma)} \le \|\tau_1\|_{H^1(B)}$ and that $\|\tau_1\|_{H^1(B)} \le \|\tau\|_{H^1(B)}$ since $\{\kappa_i\}_{i\in\mathbb{N}} \subset H^1(B)$ is an orthogonal basis. Thus the above inequality implies (after integrating in time)

$$\left\|\partial_t u^N\right\|_{L^2(0,T;(H^1(B))')} \le \left(D\left\|u^N\right\|_{L^2(0,T;H^1(B))} + \left\|q(u^N,v^N)\right\|_{L^2(0,T;L^2(\Gamma))}\right)$$

The norm $\left\|u^N\right\|_{L^2(0,T;H^1(B))}$ can be controlled directly by energy estimate (4.13) while similar arguments as in (4.11) allow us to deduce that $\left\|q(u^N,v^N)\right\|_{L^2(0,T;L^2(\Gamma))}$ is bounded by the constant $C(T)$ from (4.13). Analogously, we obtain $\left\|\partial_t\varphi^N\right\|_{L^2(0,T;H^{-1}(\Gamma))} \le C(T)$ and $\left\|\partial_t v^N\right\|_{L^2(0,T;H^{-1}(\Gamma))} \le C(T)$.

The Aubin-Lions theorem [Sim87, Corollary 2] applied for the Gelfand triple $H^1(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow H^{-1}(\Gamma)$ allows us to deduce the relative compactness of $\{v^N\}$ and $\{\varphi^N\}$ in $L^2([0,T]\times\Gamma)$ and consequently (up to the extraction of a subsequence) the convergence of $v^N(x)$ and $\varphi^N(x)$ pointwise almost everywhere in $\Gamma\times[0,T]$.

From Lemma 4.1 we finally obtain the weak convergence

$$q(u^N,v^N) \rightharpoonup q(u,v) \text{ in } L^2(0,T;L^2(\Gamma)).$$

Summing up our results, we thus deduce that there exist subsequences (which we also denote by $(u^N,\varphi^N,\mu^N,\theta^N,v^N)$ such that

$$u^N \rightharpoonup u \text{ in } L^2(0,T;H^1(B)), \tag{4.15}$$

$$u^N \to u \text{ in } L^2(0,T;H^s(B)), 0<s<1, \tag{4.16}$$

$$\mathrm{tr}(u^N) \to \mathrm{tr}(u) \text{ in } L^2(0,T;L^2(\Gamma)) \text{ and } \mathrm{tr}(u^N)(x) \to \mathrm{tr}(u)(x) \text{ a.e. in } \Gamma_T, \tag{4.17}$$

$$\varphi^N \rightharpoonup \varphi \text{ in } L^2(0,T;H^1(\Gamma)) \text{ and } \varphi^N \to \varphi \text{ in } L^2(0,T;L^2(\Gamma)), \tag{4.18}$$

$$\mu^n \rightharpoonup \mu \text{ in } L^2(0,T;H^1(\Gamma)), \tag{4.19}$$

$$\theta^n \rightharpoonup \theta \text{ in } L^2(0,T;H^1(\Gamma)), \tag{4.20}$$

$$v^N \rightharpoonup v \text{ in } L^2(0,T;H^1(\Gamma)) \text{ and } v^N \to v \text{ in } L^2(0,T;L^2(\Gamma)), \tag{4.21}$$

$$v^N(x) \to v(x) \text{ a.e. in } \Gamma_T, \tag{4.22}$$

while the time derivatives fulfil

$$\partial_t u^N \rightharpoonup \partial_t u \text{ in } L^2(0,T;\left(H^1(B)\right)'), \tag{4.23}$$

$$\partial_t\varphi^N \rightharpoonup \partial_t\varphi \text{ in } L^2(0,T;H^{-1}(\Gamma)), \tag{4.24}$$

$$\partial_t v^N \rightharpoonup \partial_t v \text{ in } L^2(0,T;H^{-1}(\Gamma)). \tag{4.25}$$

We have already seen that the non-linear exchange term term $q(u^N,v^N)$ fulfils

$$q(u^N,v^N) \rightharpoonup q(u,v) \text{ in } L^2(0,T;L^2(\Gamma)).$$

By the uniform bounds in (4.13) and the deduced convergence results in (4.18), φ^N posses a pointwise almost everywhere convergent subsequence and consequently, we have

$$W'(\varphi^N(x)) \to W'(\varphi(x)) \text{ a.e. in } \Gamma_T.$$

Furthermore, $W'(s) = 4s^3 - s$ and thus (4.18) implies that there exists a function $\chi \in L^{4/3}(\Gamma_T)$ with

$$W'(\varphi^N) \rightharpoonup \chi \text{ in } L^{4/3}(\Gamma_T).$$

Since by [DiB02, Proposition 9.1c] pointwise and weak limits must coincide, we have found $\chi = W'(\varphi)$ and

$$W'(\varphi^N) \rightharpoonup W'(\varphi) \text{ in } L^{4/3}(\Gamma_T). \tag{4.26}$$

Let $N_0 \in \mathbb{N}$ be arbitrary and consider the weak formulation of equations (2.2)–(2.7) for test functions $\omega \in C_0^1(0,T;V_\Gamma^{N_0})$ and $\kappa \in C_0^1(0,T;V_B^{N_0})$. From the convergence results in (4.16)–(4.22) and (4.23)–(4.26) we derive that the limit functions u, v, φ, μ and θ fulfil

$$\begin{aligned}
\int_0^T \langle \partial_t u, \kappa \rangle_{(H^1(B))', H^1(B)} &= -D \int_0^T \int_B \nabla u \cdot \nabla \kappa - \int_0^T \int_\Gamma q(u,v)\kappa, \\
\int_0^T \langle \partial_t \varphi, \omega \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} &= -\int_0^T \int_\Gamma \nabla_\Gamma \mu \cdot \nabla_\Gamma \omega, \\
\int_0^T \int_\Gamma \mu\omega &= \int_0^T \int_\Gamma \left[\varepsilon \nabla_\Gamma \varphi \cdot \nabla_\Gamma \omega + \varepsilon^{-1} W'(\varphi)\omega - \delta^{-1}(2v - 1 - \varphi)\omega\right], \\
\int_0^T \langle \partial_t v, \omega \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} &= -\int_0^T \int_\Gamma \nabla_\Gamma \theta \cdot \nabla_\Gamma \omega + \int_0^T \int_\Gamma q(u,v)\omega,
\end{aligned}$$

first for all $\omega \in C_0^1(0,T;V_\Gamma^{N_0})$ and $\kappa \in C_0^1(0,T;V_B^{N_0})$ but since $N_0 \in \mathbb{N}$ was arbitrary also for all $\omega \in C_0^1(0,T;\bigcup_{N\in\mathbb{N}} V_\Gamma^N)$ and $\kappa \in C_0^1(0,T;\bigcup_{N\in\mathbb{N}} V_B^N)$. Using that $\bigcup_{N\in\mathbb{N}} V_\Gamma^N$ and $\bigcup_{N\in\mathbb{N}} V_B^N$ are dense in $H^1(0,T;H^1(\Gamma))$ and $H^1(0,T;H^1(B))$ respectively, we deduce that these equations actually hold for all test functions $\omega \in H^1(0,T;H^1(\Gamma))$ and $\kappa \in H^1(0,T;H^1(B))$.

Now observe that for all $\kappa \in C^1([0,T];V_B^{N_0})$ such that $\kappa(T) = 0$

$$\begin{aligned}
&\int_B u(x,0)\kappa(x,0)\,dx \\
&= -\int_0^T \langle \partial_t u(\cdot,t), \kappa(\cdot,t) \rangle_{H^{-1}(B),H^1(B)}\,dt - \int_0^T \langle u(\cdot,t), \partial_t \kappa(\cdot,t) \rangle_{H^{-1}(B),H^1(B)}\,dt \\
&= D\int_0^T \int_B \nabla u \cdot \nabla \kappa + \int_0^T \int_\Gamma q(u,v)\kappa - \int_0^T \langle u(\cdot,t), \partial_t \kappa(\cdot,t) \rangle_{H^{-1}(B),H^1(B)}\,dt \\
&= \lim_{N\to\infty} \left(D\int_0^T \int_B \nabla u^N \cdot \nabla \kappa + \int_0^T \int_\Gamma q(u^N,v^N)\kappa - \int_0^T \left\langle u^N(\cdot,t), \partial_t \kappa(\cdot,t) \right\rangle_{H^{-1}(B),H^1(B)}\,dt \right).
\end{aligned}$$

By (4.6) we deduce that

$$\begin{aligned}
&\int_0^T \left\langle u^N(\cdot,t), \partial_t \kappa(\cdot,t) \right\rangle_{H^{-1}(B),H^1(B)}\,dt \\
&= -\int_0^T \left\langle \partial_t u^N(\cdot,t), \kappa(\cdot,t) \right\rangle_{H^{-1}(B),H^1(B)}\,dt - \left\langle u^N(\cdot,0), \kappa(\cdot,0) \right\rangle_{H^{-1}(\Gamma),H^1(\Gamma)} \\
&= D\int_0^T \int_B \nabla u^N \cdot \nabla \kappa\,dt + \int_0^T \int_\Gamma q(u^N,v^N)\kappa\,dt - \left\langle u^N(\cdot,0), \kappa(\cdot,0) \right\rangle_{H^{-1}(\Gamma),H^1(\Gamma)}.
\end{aligned}$$

Hence

$$
\begin{aligned}
&\int_B u(x,0)\kappa(x,0)\,dx \\
&= \lim_{N\to\infty}\left(D\int_0^T\int_B \nabla u^N\cdot\nabla\kappa + \int_0^T\int_\Gamma q(u^N,v^N)\kappa - \int_0^T \left\langle u^N(\cdot,t),\partial_t\kappa(\cdot,t)\right\rangle_{H^{-1}(B),H^1(B)}\,dt\right) \\
&= \lim_{N\to\infty}\left\langle u^N(\cdot,0),\kappa(\cdot,0)\right\rangle_{H^{-1}(\Gamma),H^1(\Gamma)} = \int_B u_0(x)\kappa(x,0)\,dx
\end{aligned}
$$

for $\kappa\in C^1([0,T];V_B^{N_0})$ with $\kappa(T)=0$ and $N_0\in\mathbb{N}$ arbitrary. Thus $u(\cdot,0)=u_0(\cdot)$ in $L^2(B)$. In the same way, we deduce $\varphi(\cdot,0)=\varphi_0(\cdot)$ and $v(\cdot,0)=v_0(\cdot)$ in $L^2(\Gamma)$.

Finally, (4.13) is uniform in N and therefore implies (2.27). $\square$

4.1.1 Higher Regularity for Solutions to the Full Model

Theorem 4.3 (Higher regularity)**.** *We choose again the double-well potential $W(s)=(s^2-1)^2$ in the Ginzburg-Landau part of $\mathcal{F}$ and let u_0, v_0 and φ_0 be as in Theorem 4.2. Assume that $q\in C^1(\mathbb{R}^2)$ and let $q=q(u,v)$ fulfil*

$$|q(u,v)|\le C(1+|u|+|v|).$$

In addition, we assume that

$$|D_u q(u,v)|\le C\left(1+|u|^{2/3}+|v|\right) \tag{4.27}$$

and

$$|D_v q(u,v)|\le C\left(1+|u|^{2/3}+|v|\right). \tag{4.28}$$

Finally, let (u,φ,v,μ,θ) be a weak solution to problem as obtained in Theorem 4.2. Then in fact

$$
\begin{aligned}
&u\in L^2(0,T;H^2(B)),\\
&v\in L^\infty(0,T;H^1(\Gamma))\cap L^2(0,T;H^3(\Gamma)),\\
&\varphi\in L^\infty(0,T;H^2(\Gamma))\cap L^2(0,T;H^5(\Gamma)),\\
&\mu\in L^\infty(0,T;H^3(\Gamma))\cap L^2(0,T;H^5(\Gamma)),\ \textit{and}\\
&\theta\in L^\infty(0,T;H^2(\Gamma))\cap L^2(0,T;H^3(\Gamma))
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\partial_t u\in L^\infty(0,T;L^2(B))\cap L^2(0,T;H^1(B)),\\
&\partial_t\varphi\in L^\infty(0,T;H^1(\Gamma))\cap L^2(0,T;H^3(\Gamma)),\ \textit{and}\\
&\partial_t\theta\in L^\infty(0,T;L^2(\Gamma))\cap L^2(0,T;H^1(\Gamma)).
\end{aligned}
$$

Before we prove the theorem, we state the following consequence from the growth assumptions (4.27) and (4.28) on $D_u q$ and $D_v q$.

Lemma 4.4. Let $u: B\to\mathbb{R}$ and $v:\Gamma\to\mathbb{R}$ be bounded in $L^2(0,T;H^1(B)\cap L^\infty(0,T;L^2(B))$ and in $L^2(0,T;H^1(\Gamma)\cap L^\infty(0,T;L^2(\Gamma))$ respectively and assume that q fulfils the conditions (4.27) and (4.28). Then in fact

$$D_u q(u,v), D_v q(u,v)\in L^6(0,T;L^3(\Gamma))\cap L^4(0,T;L^4(\Gamma)). \tag{4.29}$$

Proof. We only prove the assertion of the lemma for $D_u q(u,v)$ since both $D_u q(u,v)$ and $D_v q(u,v)$ fulfil the same growth property.

We start with the observation that for $s \in (0,1)$ the space $H^s(B)$ fulfils

$$\|f\|_{H^s(B)} \le C \|f\|_{L^2(B)}^{1-s} \|f\|_{H^1(B)}^s$$

for all $f \in H^1(B)$, see Remark 3.10. Lemma 3.11 thus yields for $u \in L^2(0,T;H^1(B) \cap L^\infty(0,T;L^2(B))$ and $p \ge 2$ that $u \in L^p(0,T;H^{2/p}(B))$.

For $2 \le p < 4$ the Trace Theorem 3.7 hence allows us to deduce $u \in L^p(0,T;H^{2/p-1/2}(\Gamma))$ where we use the convention $H^0(\Gamma) := L^2(\Gamma)$.

Similarly, $v \in L^2(0,T;H^1(B) \cap L^\infty(0,T;L^2(B))$ implies that $v \in L^p(0,T;H^{2/p}(\Gamma))$ for all $p \ge 2$ and in particular $v \in L^4(0,T;L^4(\Gamma)) \subset L^2(0,T;L^4(\Gamma))$ for $p = 4$ since $H^{1/2}(\Gamma) \hookrightarrow L^4(\Gamma)$.

We use this considerations to estimate

$$\begin{aligned}\int_0^T \left(\int_\Gamma |D_u q(u,v)|^4\right)^{4/4} &\le C\int_0^T \left(\int_\Gamma \left|1 + |u|^{2/3} + |v|\right|^4\right) \le C\int_0^T \left(\int_\Gamma 1 + |u|^{8/3} + |v|^4\right) \\ &\le C(\Gamma,T) + C\left(\int_0^T \left(\int_\Gamma |u|^{8/3}\right)^{\frac{8\cdot 3}{3\cdot 8}}\right) + C\left(\int_0^T \left(\int_\Gamma |v|^4\right)^{4/4}\right),\end{aligned}$$

where the last term is finite by the considerations on v above. As before, we find $u \in L^p(0,T;H^{2/p-1/2}(\Gamma))$ for $2 \le p < 4$. By the Sobolev embedding theorem, we thus have $u \in L^p(0,T;L^{\frac{4p}{3p-4}}(\Gamma))$ which for $p = \frac{8}{3}$ gives $u \in L^{8/3}(0,T;L^{8/3}(\Gamma))$. Hence the second term is finite as well, implying $D_u q(u,v) \in L^4(0,T;L^4(\Gamma))$.

Analogously, we find

$$\begin{aligned}\int_0^T \left(\int_\Gamma |D_u q(u,v)|^3\right)^{6/3} &\le C\int_0^T \left(\int_\Gamma \left|1 + |u|^{2/3} + |v|\right|^3\right)^2 \le C\int_0^T \left(\int_\Gamma 1 + |u|^2 + |v|^3\right)^2 \\ &\le C(\Gamma,T) + C\left(\int_0^T \left(\int_\Gamma |u|^2\right)^2\right) + C\left(\int_0^T \left(\int_\Gamma |v|^3\right)^2\right)\end{aligned}$$

By [LU68, Chapter 2, (2.27)] the interpolation estimate

$$\|u\|_{L^2(\Gamma)} \le C \|u\|_{H^1(B)}^{1/2} \|u\|_{L^2(B)}^{1/2}$$

holds. Integrating $\|u(t)\|_{L^2(\Gamma)}^4$ in time thus yields

$$\|u\|_{L^4(0,T;L^2(\Gamma))} \le C \|u\|_{L^\infty(0,T;L^2(B))} \|u\|_{L^2(0,T;H^1(B))},$$

which is bounded for $u \in L^2(0,T;H^1(B) \cap L^\infty(0,T;L^2(B))$. Therefore the second term on the right hand-side in the foregoing estimate is finite. As above, $v \in L^p(0,T;H^{2/p}(\Gamma))$ for all $p \ge 2$ and in particular for $p = 6$. By the Sobolev embedding theorem we have $H^{1/3}(\Gamma) \hookrightarrow L^3(\Gamma)$ and thus the third term above is finite. Altogether, we obtain $D_u q(u,v) \in L^6(0,T;L^3(\Gamma))$. □

Proof of Theorem 4.3. The proof of Theorem 4.3 can be divided into three steps. In the first step, we consider the approximate solutions $(u^N, v^N, \varphi^N, \mu^N, \theta^N)$ from the proof of the existence theorem (Theorem 4.2) and prove regularity estimates for these functions and their time derivatives. Secondly, we show that the limit functions of these time derivatives as $N \to \infty$ converge to solutions of the linearised model. This step is summarized in Lemma 4.6. The main tool for the proof is Lemma 4.5 from the first step. Finally, we derive higher regularity for the full system from the additional information gathered from the linearised system.

First Step: Higher regularity for the approximate solutions. We recall the proof of Theorem 4.2 and let $(u^N, v^N, \varphi^N, \mu^N, \theta^N)$ denote the subsequence of solutions to the approximate problem (4.6)–(4.9) which converges to $(u, \varphi, v, \mu, \theta)$. Let P_N^Γ denote the orthogonal projection in $H^1(\Gamma)$ onto V_Γ^N, where V_Γ^N is defined as in the proof of Theorem 4.2. Thus φ^N, μ^N and $\theta^N \in V_\Gamma^N$ fulfil

$$\int_\Gamma \mu^N \omega = \varepsilon \int_\Gamma \nabla_\Gamma \varphi^N \cdot \nabla_\Gamma \omega + \frac{1}{\varepsilon} \int_\Gamma P_N^\Gamma W'(\varphi^N)\omega - \int_\Gamma \frac{\theta^N}{2}\omega$$

for all $\omega \in V_\Gamma^N$. By the orthogonal decomposition $H^1(\Gamma) = V_\Gamma^N \oplus \left(V_\Gamma^N\right)^\perp$ this equation also holds for all test functions $\omega \in H^1(\Gamma)$, which implies that φ^N is a weak solution to the elliptic equation

$$-\varepsilon \Delta_\Gamma \varphi^N = \mu^N + \frac{\theta^N}{2} - \frac{1}{\varepsilon} P_N^\Gamma W'(\varphi^N). \tag{4.30}$$

Furthermore, the energy estimate (4.13) together with (4.14) yields

$$\mu^N, \theta^N \in L^2(0,T; H^1(\Gamma)) \text{ and } \varphi^N \in L^\infty(0,T; H^1(\Gamma)).$$

In particular,

$$\varphi^N \in L^\infty(0,T; L^p(\Gamma)) \text{ for all } 1 \le p < \infty \tag{4.31}$$

by the the Sobolev embedding theorem in dimension $\dim \Gamma = 2$.

Observe that therefore every polynomial in φ^N is an element of $L^\infty(0,T; L^p(\Gamma))$ for all $1 \le p < \infty$. We will exploit this property in particular with respect to $W'(\varphi^N), W''(\varphi^N)$, and $W'''(\varphi^N)$ since these terms grow at most polynomial in φ^N. For example, $W'(\varphi^N)$ fulfils $\left|W'(\varphi^N)\right| \le C(\left|\varphi^N\right|^3 + 1)$ for some $C > 0$.

As a first application, we directly deduce the boundedness of $W'(\varphi^N)$ in $L^2(0,T; L^2(\Gamma))$. Hence the right hand-side in (4.30) is in $L^2(0,T; L^2(\Gamma))$. Elliptic theory, see e.g. [GT01, Theorem 8.8, Theorem 8.12], thus implies that the solution φ^N to (4.30) fulfils $\varphi^N \in L^2(0,T; H^2(\Gamma))$. We remark that all these estimates are derived from the energy estimate (4.13), which is uniform in N. Hence we conclude that $\{\varphi^N\}_{N\in\mathbb{N}} \subset L^2(0,T; H^2(\Gamma))$ is uniformly bounded in N.

Additionally, the Sobolev embedding and $\varphi^N \in L^2(0,T : H^2(\Gamma))$ directly yield

$$\left\|\varphi^N\right\|_{L^2(0,T;W^{1,p}(\Gamma))} \le C \text{ for all } 1 \le p < \infty. \tag{4.32}$$

We calculate

$$\begin{aligned}
\int_0^T \int_\Gamma \left|\nabla_\Gamma \left(W'(\varphi^N)\right)\right|^2 &\le C \int_0^T \int_\Gamma \left|W''(\varphi^N)\right|^2 \left|\nabla_\Gamma \varphi^N\right|^2 \\
&\le \int_0^T \left(\int_\Gamma \left(\left|W''(\varphi^N)\right|\right)^4\right)^{1/2} \left(\int_\Gamma \left|\nabla_\Gamma \varphi^N\right|^4\right)^{1/2} \\
&\le C \left(\left\|W''(\varphi^N)\right\|_{L^\infty(0,T;L^4(\Gamma))} + 1\right) \left\|\nabla_\Gamma \varphi^N\right\|_{L^2(0,T;L^4(\Gamma))},
\end{aligned}$$

which yields a uniform bound in N for $\left\|W'(\varphi^N)\right\|_{L^2(0,T;H^1(\Gamma))}$ by (4.13) and the foregoing discussion. Moreover, $\left\|P_N^\Gamma\right\|_{\mathcal{L}(H^1(\Gamma),H^1(\Gamma))} \le 1$ implies

$$\left\|P_N^\Gamma W'(\varphi^N)\right\|_{L^2(0,T;H^1(\Gamma))} \le \left\|W'(\varphi^N)\right\|_{L^2(0,T;H^1(\Gamma))},$$

showing that the right hand side in (4.30) belongs to $L^2(0,T;H^1(\Gamma))$. As a direct consequence, we infer

$$\varphi^N \in L^2(0,T;H^3(\Gamma)) \cap L^\infty(0,T;H^1(\Gamma)),$$

by using standard elliptic theory, see e.g. [GT01, Theorem 8.8, Theorem 8.12]. Note that the bound in $L^2(0,T;H^1(\Gamma))$ on the right hand side in (4.30) is uniform in N and hence we also have

$$\|\varphi^N\|_{L^2(0,T;H^3(\Gamma))\cap L^\infty(0,T;H^1(\Gamma)))} \le C$$

uniformly in N.

We remark for later use that the same argument applied to the equation (2.5) also implies

$$\varphi \in L^2(0,T;H^3(\Gamma)) \cap L^\infty(0,T;H^1(\Gamma)) \hookrightarrow L^2(0,T;W^{2,p}(\Gamma)) \text{ for all } 1 \le p < \infty. \tag{4.33}$$

Next we differentiate the equations (4.6)–(4.9) in time. Note that the approximate solutions $u^N, \varphi^N, v^N, \mu^N$, and θ^N were all constructed from solutions to a system of ordinary differential solutions, i.e. they are all differentiable in t. We introduce the notation

$$\tilde{u}^N = \partial_t u^N, \tilde{\varphi}^N = \partial_t \varphi^N, \tilde{v}^N = \partial_t v^N, \tilde{\mu}^N = \partial_t \mu^N \text{ and } \tilde{\theta}^N = \partial_t \theta^N.$$

The tuple $(\tilde{u}^N, \tilde{\varphi}^N, \tilde{v}^N, \tilde{\mu}^N, \tilde{\theta}^N)$ solves for all $\kappa \in V_B^N$ and all $\omega \in V_\Gamma^N$

$$\int_B \partial_t \tilde{u}^N \kappa = -D \int_B \nabla \tilde{u}^N \cdot \nabla \kappa - \int_\Gamma \frac{d}{dt}\left(q(u^N, v^N)\right)\kappa, \tag{4.34}$$

$$\int_\Gamma \partial_t \tilde{\varphi}^N \omega = -\int_\Gamma \nabla_\Gamma \tilde{\mu}^N \cdot \nabla_\Gamma \omega \tag{4.35}$$

$$\int_\Gamma \tilde{\mu}^N \omega = \int_\Gamma \left[\varepsilon \nabla_\Gamma \tilde{\varphi}^N \cdot \nabla_\Gamma \omega + \varepsilon^{-1} W''(\varphi^N)\tilde{\varphi}^N \omega - \frac{\tilde{\theta}^N}{2}\omega\right] \tag{4.36}$$

$$\int_\Gamma \tilde{\theta}^N \omega = \frac{2}{\delta}\int_\Gamma (2\tilde{v}^N - \tilde{\varphi}^N)\omega \tag{4.37}$$

$$\int_\Gamma \frac{\delta}{4}\partial_t \tilde{\theta}^N \omega + \int_\Gamma \frac{1}{2}\partial_t \tilde{\varphi}^N \omega = -\int_\Gamma \nabla_\Gamma \tilde{\theta}^N \cdot \nabla_\Gamma \omega + \int_\Gamma \frac{d}{dt}\left(q(u^N, v^N)\right)\omega. \tag{4.38}$$

Lemma 4.5. Let $(\tilde{u}^N, \tilde{\varphi}^N, \tilde{v}^N, \tilde{\mu}^N, \tilde{\theta}^N)$ be defined as above. Under the assumptions of Theorem 4.3 the estimate

$$\begin{aligned}\sup_{t\in(0,T)} &\left\{\frac{\varepsilon}{2}\|\nabla_\Gamma \tilde{\varphi}^N\|^2_{L^2(\Gamma)} + \frac{\delta}{8}\|\tilde{\theta}^N\|^2_{L^2(\Gamma)} + \frac{1}{2}\|\tilde{u}^N\|^2_{L^2(B}\right\} \\ &+ \int_\Gamma |\nabla_\Gamma \tilde{\mu}^N|^2 + \int_\Gamma |\nabla_\Gamma \tilde{\theta}^N|^2 + D\int_B |\nabla \tilde{u}^N|^2 \le C(T).\end{aligned} \tag{4.39}$$

holds. The estimate is uniform in N.

Proof of Lemma 4.5. We choose $\omega = \tilde{u}^N$ as a test function in (4.34), $\omega = \tilde{\mu}^N$ in (4.35), $\omega = \partial_t \tilde{\varphi}^N$ in (4.36) and $\omega = \tilde{\theta}^N$ in (4.38). We add these equations to deduce

$$\begin{aligned}\frac{\varepsilon}{2}\frac{d}{dt}\int_\Gamma |\nabla_\Gamma \tilde{\varphi}^N|^2 + \frac{\delta}{8}\frac{d}{dt}\int_\Gamma |\tilde{\theta}^N|^2 + \int_\Gamma |\nabla_\Gamma \tilde{\mu}^N|^2 + \int_\Gamma |\nabla_\Gamma \tilde{\theta}^N|^2 + \frac{1}{2}\frac{d}{dt}\int_B |\tilde{u}^N|^2 + D\int_B |\nabla \tilde{u}^N|^2 \\ = -\frac{1}{\varepsilon}\int_\Gamma W''(\varphi^N)\tilde{\varphi}^N \partial_t \tilde{\varphi}^N + \int_\Gamma \frac{d}{dt}\left(q(u^N, v^N)\right)\left(\tilde{\theta}^N - \tilde{u}^N\right).\end{aligned} \tag{4.40}$$

To estimate the right hand side in (4.40) we first compute for any $\gamma > 0$

$$\begin{aligned}\left|\frac{1}{\varepsilon}\int_\Gamma W''(\varphi^N)\tilde{\varphi}^N\partial_t\tilde{\varphi}^N\right| &= \left|\frac{1}{\varepsilon}\int_\Gamma \nabla_\Gamma\left(W''(\varphi^N)\tilde{\varphi}^N\right)\cdot\nabla_\Gamma\tilde{\mu}^N\right| \\ &\le \frac{C_\gamma}{\varepsilon}\int_\Gamma\left|\nabla_\Gamma\left(W''(\varphi^N)\tilde{\varphi}^N\right)\right|^2 + \frac{\gamma}{\varepsilon}\int_\Gamma\left|\nabla_\Gamma\tilde{\mu}^N\right|^2 \end{aligned} \tag{4.41}$$

where we have used that $\partial_t\tilde{\varphi}^N = \Delta_\Gamma\tilde{\mu}^N$ almost everywhere since by definition we have $\tilde{\varphi}^N \in V_\Gamma^N$ and $\tilde{\mu}^N \in V_\Gamma^N$ for all $t \in (0,T)$, i.e. (4.35) implies for all $t \in (0,T)$ the identity $\partial_t\tilde{\varphi}^N = \Delta_\Gamma\tilde{\mu}^N$ in V_Γ^N and thus $\partial_t\tilde{\varphi}^N = \Delta_\Gamma\tilde{\mu}^N$ almost everywhere in Γ_T.

The first term on the right hand side can be controlled by $\int_\Gamma\left|\nabla_\Gamma\tilde{\varphi}^N\right|^2$ in the following way. By the growth properties of W, we have

$$\begin{aligned}\int_\Gamma\left|\nabla_\Gamma\left(W''(\varphi^N)\tilde{\varphi}^N\right)\right|^2 &\le 2\int_\Gamma\left|W''(\varphi^N)\right|^2\left|\nabla_\Gamma\tilde{\varphi}^N\right|^2 + 2\int_\Gamma\left|\nabla_\Gamma\left(W''(\varphi^N)\right)\right|^2\left|\tilde{\varphi}^N\right|^2 \\ &\le C\left(\|\varphi^N(t)\|^2_{L^\infty(\Gamma)}+1\right)\left(\int_\Gamma\left|\nabla_\Gamma\tilde{\varphi}^N\right|^2\right) \\ &\quad + 2\int_\Gamma\left|\nabla_\Gamma\left(W''(\varphi^N)\right)\right|^2\left|\tilde{\varphi}^N\right|^2. \end{aligned} \tag{4.42}$$

Moreover, we apply Hölder's inequality to deduce

$$\begin{aligned}\int_\Gamma\left|\nabla_\Gamma\left(W''(\varphi^N)\right)\right|^2\left|\tilde{\varphi}^N\right|^2 &\le C\int_\Gamma\left|\varphi^N+1\right|^2\left|\nabla_\Gamma\varphi^N\right|^2\left|\tilde{\varphi}^N\right|^2 \\ &\le C\left(\int_\Gamma\left|\varphi^N+1\right|^3\left|\nabla_\Gamma\varphi^N\right|^3\right)^{2/3}\left(\int_\Gamma\left|\tilde{\varphi}^N\right|^6\right)^{1/3} \\ &\le C\left(\int_\Gamma\left|\varphi^N+1\right|^6\right)^{2/6}\left(\int_\Gamma\left|\nabla_\Gamma\varphi^N\right|^6\right)^{2/6}\left(\int_\Gamma\left|\tilde{\varphi}^N\right|^6\right)^{1/3}. \end{aligned}$$

Using that $\int_\Gamma\tilde{\varphi}^N = \int_\Gamma\Delta_\Gamma\mu^N = 0$ we have furthermore

$$\left(\int_\Gamma\left|\tilde{\varphi}^N\right|^6\right)^{1/3} \le C\left(\int_\Gamma\left|\nabla_\Gamma\tilde{\varphi}^N\right|^2\right)$$

by the Sobolev embedding theorem, i.e.

$$\int_\Gamma\left|\nabla_\Gamma\left(W''(\varphi^N)\right)\right|^2\left|\tilde{\varphi}^N\right|^2 \le C\left(\int_\Gamma\left|\varphi^N+1\right|^6\right)^{2/6}\left(\int_\Gamma\left|\nabla_\Gamma\varphi^N\right|^6\right)^{2/6}\left(\int_\Gamma\left|\nabla_\Gamma\tilde{\varphi}^N\right|^2\right).$$

Thus (4.42) reads

$$\begin{aligned}\int_\Gamma\left|\nabla_\Gamma\left(W''(\varphi^N)\tilde{\varphi}^N\right)\right|^2 \le& 2\int_\Gamma\left|W''(\varphi^N)\right|^2\left|\nabla_\Gamma\tilde{\varphi}^N\right|^2 + 2\int_\Gamma\left|\nabla_\Gamma\left(W''(\varphi^N)\right)\right|^2\left|\tilde{\varphi}^N\right|^2 \\ \le& C\left(\|\varphi^N(t)\|^4_{L^\infty(\Gamma)}+1\right)\left(\int_\Gamma\left|\nabla_\Gamma\tilde{\varphi}^N\right|^2\right) \\ &+ C\left(\int_\Gamma\left|\varphi^N+1\right|^6\right)^{2/6}\left(\int_\Gamma\left|\nabla_\Gamma\varphi^N\right|^6\right)^{2/6}\left(\int_\Gamma\left|\nabla_\Gamma\tilde{\varphi}^N\right|^2\right). \end{aligned} \tag{4.43}$$

We remark that $\left(\|\varphi^N(t)\|^4_{L^\infty(\Gamma)} + 1\right)$ is bounded in $L^1(0,T)$ by the following argument. Recall that $H^{3/2}(\Gamma) = (H^1(\Gamma), H^2(\Gamma))_{1/2,2}$ as discussed in Remark 3.10. Since $\varphi^N \in L^\infty(0,T;H^1(\Gamma))$ and $\varphi^N \in L^2(0,T;H^2(\Gamma))$, Lemma 3.11 implies $\varphi^N \in L^4\left(0,T;H^{3/2}(\Gamma)\right)$. Hence the embedding

$$\left(H^1(\Gamma);H^2(\Gamma)\right)_{1/2,2} = H^{3/2}(\Gamma) \hookrightarrow C^{0,\alpha}(\Gamma) \text{ for } 0 < \alpha < 1/2$$

yields $\varphi^N \in L^4(0,T;L^\infty(\Gamma))$. Likewise, (4.31) and (4.32) imply

$$\left(\int_\Gamma \left|\varphi^N(t)+1\right|^6\right)^{2/6} \in L^\infty(0,T) \quad \text{and} \quad \left(\int_\Gamma \left|\nabla_\Gamma \varphi^N(t)\right|^6\right)^{2/6} \in L^1(0,T)$$

uniformly in N, from which we deduce that

$$\left(\int_\Gamma \left|\varphi^N(t)+1\right|^6\right)^{2/6} \left(\int_\Gamma \left|\nabla_\Gamma \varphi^N(t)\right|^6\right)^{2/6} \in L^1(0,T).$$

Hence

$$F^N(t) := \max\left\{\left(\int_\Gamma \left|\varphi^N(t)+1\right|^6\right)^{2/6} \left(\int_\Gamma \left|\nabla_\Gamma \varphi^N(t)\right|^6\right)^{2/6}, \left(\|\varphi^N(t)\|^4_{L^\infty(\Gamma)} + 1\right)\right\} \in L^1(0,T)$$

and there exists a constant $C > 0$ such that

$$\|F^N\|_{L^1(0,T)} \le C$$

uniformly in N.

Combining (4.41) and (4.43) we arrive at

$$\left|\frac{1}{\varepsilon}\int_\Gamma W''(\varphi^N)\tilde{\varphi}^N \partial_t \tilde{\varphi}^N\right| \le \frac{2C_\gamma}{\varepsilon} F^N(t) \left(\int_\Gamma \left|\nabla_\Gamma \tilde{\varphi}^N\right|^2\right) + 2\frac{\gamma}{\varepsilon}\int_\Gamma \left|\nabla_\Gamma \tilde{\mu}^N\right|^2. \tag{4.44}$$

We have thus estimated the first term on the right hand-side in (4.40) and it remains to control the second term on the right hand-side in this inequality. To this end, we compute

$$\begin{aligned}
&\left|\int_\Gamma \frac{d}{dt}\left(q(u^N,v^N)\right)\left(\tilde{\theta}^N - \tilde{u}^N\right)\right| \\
&\le \int_\Gamma \left|D_u q(u^N,v^N)\right|\left|\tilde{u}^N\right|^2 + \int_\Gamma \left|D_u q(u^N,v^N)\right|\left|\tilde{u}^N\right|\left|\tilde{\theta}^N\right| \\
&\quad + \int_\Gamma \left|D_v q(u^N,v^N)\right|\left|\tilde{v}^N\right|\left|\tilde{u}^N\right| + \int_\Gamma \left|D_v q(u^N,v^N)\right|\left|\tilde{v}^N\right|\left|\tilde{\theta}^N\right|
\end{aligned} \tag{4.45}$$

In order to shorten the estimate for the last three terms, let f, g, h be measurable functions on Γ. We deduce for all $\gamma > 0$

$$\begin{aligned}
\int_\Gamma |f||g||h| &\le \|f\|_{L^4(\Gamma)} \|g\|_{L^2(\Gamma)} \|h\|_{L^4(\Gamma)} \\
&\le C_\gamma \|f\|^2_{L^4(\Gamma)} \|g\|^2_{L^2(\Gamma)} + \gamma \|h\|^2_{L^4(\Gamma)}
\end{aligned} \tag{4.46}$$

from Young's inequality, where we used the generalized Hölder inequality in the first step.

We remark that using the Sobolev embedding theorem and the Trace Theorem 3.7 we can always estimate

$$\left\|\tilde{u}^N\right\|^2_{L^4(\Gamma)} \leq C\left\|\tilde{u}^N\right\|_{H^{1/2}(\Gamma)} \leq C\left\|\tilde{u}^N\right\|_{H^1(B)}.$$

Moreover, $\tilde{v}^N = \frac{\delta}{2}\tilde{\theta}^N + \frac{1}{2}\tilde{\varphi}^N$ and thus by Poincaré's inequality

$$\left\|\tilde{v}^N\right\|_{L^2(\Gamma)} \leq \frac{\delta}{2}\left\|\tilde{\theta}^N\right\|_{L^2(\Gamma)} + \frac{1}{2}\left\|\tilde{\varphi}^N\right\|_{L^2(\Gamma)} \leq \frac{\delta}{2}\left\|\tilde{\theta}^N\right\|_{L^2(\Gamma)} + \frac{C}{2}\left\|\nabla_\Gamma\tilde{\varphi}^N\right\|_{L^2(\Gamma)}. \tag{4.47}$$

Choosing $f = D_v q(u^N, v^N), g = \tilde{v}^N, h = \tilde{u}^N$ and $f = D_u q(u^N, v^N), g = \tilde{\theta}^N, h = \tilde{u}^N$ respectively in (4.46), we deduce

$$\begin{aligned}
&\int_\Gamma \left|D_u q(u^N, v^N)\right|\left|\tilde{u}^N\right|\left|\tilde{\theta}^N\right| + \int_\Gamma \left|D_v q(u^N, v^N)\right|\left|\tilde{v}^N\right|\left|\tilde{u}^N\right| \\
\leq& C_\gamma\left(\left\|D_u q(u^N, v^N)\right\|^2_{L^4(\Gamma)} + \frac{\delta}{2}\left\|D_v q(u^N, v^N)\right\|^2_{L^4(\Gamma)}\right)\left\|\tilde{\theta}^N\right\|^2_{L^2(\Gamma)} \\
&+ \frac{C_\gamma}{2}\left\|D_v q(u^N, v^N)\right\|^2_{L^4(\Gamma)}\left\|\nabla_\Gamma\tilde{\varphi}^N\right\|^2_{L^2(\Gamma)} + \gamma C\left(\left\|\tilde{u}^N\right\|^2_{L^2(B)} + \left\|\nabla\tilde{u}^N\right\|^2_{L^2(B)}\right).
\end{aligned} \tag{4.48}$$

Note that we used (4.47) to estimate $\left\|\tilde{v}^N\right\|_{L^2(\Gamma)}$.

Now we choose $f = D_v q(u^N, v^N), g = \tilde{v}^N, h = \theta$ in (4.46) to obtain

$$\begin{aligned}
\int_\Gamma \left|D_v q(u^N, v^N)\right|\left|\tilde{v}^N\right|\left|\tilde{\theta}^N\right| \leq C_\gamma\left\|D_v q(u^N, v^N)\right\|^2_{L^4(\Gamma)}\left(\frac{\delta}{2}\left\|\tilde{\theta}^N\right\|^2_{L^2(\Gamma)} + \frac{1}{2}\left\|\nabla_\Gamma\tilde{\varphi}^N\right\|^2_{L^2(\Gamma)}\right) \\
+ \gamma C\left(\left\|\tilde{\theta}^N\right\|^2_{L^2(\Gamma)} + \left\|\nabla_\Gamma\tilde{\theta}^N\right\|^2_{L^2(\Gamma)}\right).
\end{aligned} \tag{4.49}$$

Finally, we use again the trace and Sobolev embedding theorems together with the interpolation inequality (3.1) to estimate

$$\begin{aligned}
\int_\Gamma \left|D_u q(u^N, v^N)\right|\left|\tilde{u}^N\right|^2 \leq& \left\|D_u q(u^N, v^N)\right\|_{L^3(\Gamma)}\left\|\tilde{u}^N\right\|^2_{L^3(\Gamma)} \\
\leq& C\left\|D_u q(u^N, v^N)\right\|_{L^3(\Gamma)}\left\|\tilde{u}^N\right\|^2_{H^{1/3}(\Gamma)} \\
\leq& C\left\|D_u q(u^N, v^N)\right\|_{L^3(\Gamma)}\left\|\tilde{u}^N\right\|^2_{H^{5/6}(B)} \\
\leq& C\left\|D_u q(u^N, v^N)\right\|_{L^3(\Gamma)}\left\|\tilde{u}^N\right\|^{1/3}_{L^2(B)}\left\|\tilde{u}^N\right\|^{5/3}_{H^1(B)} \\
\leq& C_\gamma\left\|D_u q(u^N, v^N)\right\|^6_{L^3(\Gamma)}\left\|\tilde{u}^N\right\|^2_{L^2(B)} + \gamma\left\|\tilde{u}^N\right\|^2_{H^1(B)}.
\end{aligned} \tag{4.50}$$

To simplify the notation, we introduce

$$M^N(t) = \max\left\{\left\|D_u q(u^N, v^N)\right\|^6_{L^3(\Gamma)}, \left\|D_u q(u^N, v^N)\right\|^2_{L^4(\Gamma)}, \left\|D_v q(u^N, v^N)\right\|^2_{L^4(\Gamma)}\right\}.$$

The functions u^N and v^N fulfil the assumptions of Lemma 4.4 by (4.13). Hence (4.29), and (4.29) imply $M^N(t) \in L^1(0,T)$. Moreover, the bound on M^N in $L^1(0,T)$ is uniform in N since it is derived from the uniform estimate (4.13).

We combine (4.45), (4.48), (4.49), and (4.50) and obtain

$$\begin{aligned}
&\left|\int_\Gamma \frac{d}{dt}\left(q(u^N, v^N)\right)\left(\tilde{\theta}^N - \tilde{u}^N\right)\right| \\
\leq& C_\gamma\left(M^N(t) + 1\right)\left\|\tilde{u}^N\right\|^2_{L^2(B)} + C_\gamma\left((1+\delta)M^N(t) + 1\right)\left\|\tilde{\theta}^N\right\|^2_{L^2(\Gamma)} \\
&+ C_\gamma M^N(t)\left\|\nabla_\Gamma\tilde{\varphi}^N\right\|^2_{L^2(\Gamma)} + \gamma C\left\|\nabla_\Gamma\tilde{\theta}^N\right\|^2_{L^2(\Gamma)} + \gamma C\left\|\nabla\tilde{u}^N\right\|^2_{L^2(B)},
\end{aligned} \tag{4.51}$$

which controls the second term on the right hand-side in (4.40). We thus return to (4.40) and use (4.44) and (4.51) to deduce

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\int_\Gamma \left|\nabla_\Gamma \tilde{\varphi}^N\right|^2 + \frac{\delta}{8}\frac{d}{dt}\int_\Gamma \left|\tilde{\theta}^N\right|^2 + \int_\Gamma \left|\nabla_\Gamma \tilde{\mu}^N\right|^2 + \int_\Gamma \left|\nabla_\Gamma \tilde{\theta}^N\right|^2 \\
&\qquad + \frac{1}{2}\frac{d}{dt}\int_B \left|\tilde{u}^N\right|^2 + D\int_B \left|\nabla \tilde{u}^N\right|^2 \\
&\qquad \le C_\gamma \left(M^N(t)+1\right)\left\|\tilde{u}^N\right\|^2_{L^2(B)} + C_\gamma \left((1+\delta)M^N(t)+1\right)\left\|\tilde{\theta}^N\right\|^2_{L^2(\Gamma)} \\
&\qquad + C_\gamma \left(M^N(t) + \frac{2F^N(t)}{\varepsilon}\right)\left\|\nabla_\Gamma \tilde{\varphi}^N\right\|^2_{L^2(\Gamma)} + \gamma C\left\|\nabla_\Gamma \tilde{\theta}^N\right\|^2_{L^2(\Gamma)} \\
&\qquad + \gamma C\left\|\nabla \tilde{u}^N\right\|^2_{L^2(B)} + 2\frac{\gamma}{\varepsilon}\int_\Gamma \left|\nabla_\Gamma \tilde{\mu}^N\right|^2 .
\end{aligned}$$

By taking γ to be sufficiently small, we can absorb the gradient terms on the right hand-side and conclude

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\int_\Gamma \left|\nabla_\Gamma \tilde{\varphi}^N\right|^2 + \frac{\delta}{8}\frac{d}{dt}\int_\Gamma \left|\tilde{\theta}^N\right|^2 + \int_\Gamma \left|\nabla_\Gamma \tilde{\mu}^N\right|^2 + \int_\Gamma \left|\nabla_\Gamma \tilde{\theta}^N\right|^2 \\
&\qquad + \frac{1}{2}\frac{d}{dt}\int_B \left|\tilde{u}^N\right|^2 + D\int_B \left|\nabla \tilde{u}^N\right|^2 \\
&\qquad \le C_\gamma \left(M^N(t)+1\right)\left\|\tilde{u}^N\right\|^2_{L^2(B)} + C_\gamma \left((1+\delta)M^N(t)+1\right)\left\|\tilde{\theta}^N\right\|^2_{L^2(\Gamma)} \\
&\qquad + C_\gamma \left(M^N(t) + \frac{2F^N(t)}{\varepsilon}\right)\left\|\nabla_\Gamma \tilde{\varphi}^N\right\|^2_{L^2(\Gamma)} .
\end{aligned}$$

Because of $M^N(t) \in L^1(0,T)$ and $F^N(t) \in L^1(0,T)$ uniformly in N, Gronwall's inequality yields (4.39). □

Second step: Taking the limit $N \to \infty$. Estimate (4.39) is uniform in N and allows to extract weakly converging subsequences, which for convenience we denote again by $\tilde{u}^N, \tilde{\varphi}^N, \tilde{\mu}^N$ and $\tilde{\theta}^N$. Hence there exist functions

$$\begin{aligned}
&\tilde{u} \in L^\infty(0,T;L^2(B)) \cap L^2(0,T;H^1(B)),\\
&\tilde{\varphi} \in L^\infty(0,T;H^1(\Gamma)),\\
&\tilde{\theta} \in L^\infty(0,T;L^2(\Gamma)) \cap L^2(0,T;H^1(\Gamma)) \text{ and}\\
&\tilde{\mu} \in L^2(0,T;H^1(\Gamma))
\end{aligned}$$

such that

$$\tilde{u}^N \rightharpoonup \tilde{u} \text{ in } L^2(0,T;H^1(B)), \tag{4.52}$$
$$\tilde{\varphi}^N \rightharpoonup \tilde{\varphi} \text{ in } L^2(0,T;H^1(\Gamma)), \tag{4.53}$$
$$\tilde{\theta}^N \rightharpoonup \tilde{\theta} \text{ in } L^2(0,T;H^1(\Gamma)) \text{ and} \tag{4.54}$$
$$\tilde{\mu}^N \rightharpoonup \tilde{\mu} \text{ in } L^2(0,T;H^1(\Gamma)). \tag{4.55}$$

We remark that these convergences allow us to conclude

$$\partial_t u = \tilde{u} \text{ etc.}$$

in the sense of distributions.

Lemma 4.6. The tuple $(\tilde{u}, \tilde{\varphi}, \tilde{v}, \tilde{\mu}, \tilde{\theta})$ is a weak solution to

$$\begin{aligned}
\partial_t \tilde{u} &= D\Delta\tilde{u} && \text{in } B\times(0,T], && (4.56)\\
-D\nabla\tilde{u}\cdot\nu &= D_u q(u,v)\tilde{u} + D_v q(u,v)\tilde{v} && \text{on } \Gamma\times(0,T], && (4.57)\\
\partial_t\tilde{\varphi} &= \Delta_\Gamma\tilde{\mu} && \text{on } \Gamma\times(0,T], && (4.58)\\
\tilde{\mu} &= -\varepsilon\Delta_\Gamma\tilde{\varphi} + \varepsilon^{-1}W''(\varphi)\tilde{\varphi} - \frac{1}{2}\tilde{\theta} && \text{on } \Gamma\times(0,T], && (4.59)\\
\frac{\delta}{4}\partial_t\tilde{\theta} &= \Delta_\Gamma\tilde{\theta} - \frac{1}{2}\Delta_\Gamma\tilde{\mu} + D_u q(u,v)\tilde{u} + D_v q(u,v)\tilde{v} && \text{on } \Gamma\times(0,T] && (4.60)\\
\tilde{\theta} &= \frac{2}{\delta}(2\tilde{v} - \tilde{\varphi}) && \text{on } \Gamma\times(0,T]. && (4.61)
\end{aligned}$$

Proof of Lemma 4.6. We first observe that (4.34) implies a bound on $\|\partial_t\tilde{u}^N\|_{L^2(0,T;H^{-1}(B))}$ in the following way. Let $\kappa \in L^2(0,T;H^1(B))$ and denote by P_N^B the orthogonal projection in $H^1(B)$ onto V_B^N. Then

$$\begin{aligned}
&\left|\int_0^T \langle\partial_t\tilde{u}^N, \kappa\rangle_{(H^1(B))', H^1(B)}\right| \le \int_0^T\int_B \left|\nabla\tilde{u}^N\cdot\nabla P_N^B\kappa\right| + \int_0^T\int_\Gamma\left|\frac{d}{dt}q(u^N,v^N)P_N^B\kappa\right|\\
&\le \left\|\nabla\tilde{u}^N\right\|_{L^2(0,T;L^2(B))}\|\kappa\|_{L^2(0,T;H^1(B))} + \int_0^T\int_\Gamma\left|D_u q(u^N,v^N)\tilde{u}^N P_N^B\kappa\right|\\
&\qquad + \int_0^T\int_\Gamma\left|D_v q(u^N,v^N)\tilde{v}^N P_N^B\kappa\right|. \qquad (4.62)
\end{aligned}$$

The first term is bounded by (4.39) from Lemma 4.5. The second term can be estimated by

$$\begin{aligned}
\int_0^T\int_\Gamma\left|D_u q(u^N,v^N)\tilde{u}^N P_N^B\kappa\right| &\le \int_0^T\left(\int_\Gamma\left|D_u q(u^N,v^N)\tilde{u}^N\right|^{4/3}\right)^{3/4}\left(\int_\Gamma\left|P_N^B\kappa\right|^4\right)^{1/4}\\
&\le\left(\int_0^T\left(\int_\Gamma\left|D_u q(u^N,v^N)\tilde{u}^N\right|^{4/3}\right)^{3/2}\right)^{1/2}\left(\int_0^T\left(\int_\Gamma\left|P_N^B\kappa\right|^4\right)^{1/2}\right)^{1/2}\\
&\le\left\|D_u q(u^N,v^N)\tilde{u}^N\right\|_{L^2(0,T;L^{4/3}(\Gamma))}\left\|P_N^B\kappa\right\|_{L^2(0,T;L^4(\Gamma))}\\
&\le C\left\|D_u q(u^N,v^N)\tilde{u}^N\right\|_{L^2(0,T;L^{4/3}(\Gamma))}\|\kappa\|_{L^2(0,T;H^1(B))}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left\|D_u q(u^N,v^N)\tilde{u}^N\right\|^2_{L^2(0,T;L^{4/3}(\Gamma))} &= \int_0^T\left(\int_\Gamma\left|D_u q(u^N,v^N)\tilde{u}^N\right|^{4/3}\right)^{3/2}\\
&\le\int_0^T\left(\int_\Gamma\left|D_u q(u^N,v^N)\right|^3\right)^{2/3}\left(\int_\Gamma\left|\tilde{u}^N\right|^{12/5}\right)^{5/6}\\
&\le\left(\int_0^T\left(\int_\Gamma\left|D_u q(u^N,v^N)\right|^3\right)^{6/3}\right)^{2/6}\left(\int_0^T\left(\int_\Gamma\left|\tilde{u}^N\right|^{12/5}\right)^{15/12}\right)^{2/3}\\
&=\left\|D_u q(u^N,v^N)\right\|^2_{L^6(0,T;L^3(\Gamma))}\left\|\tilde{u}^N\right\|^2_{L^3(0,T;L^{12/5}(\Gamma))},
\end{aligned}$$

where the first term is bounded by Lemma 4.4 and the second term is bounded because analogue to $u \in L^p(0,T;L^{\frac{4p}{3p-4}}(\Gamma))$ in the proof of Lemma 4.4 we obtain $\tilde{u}^N \in L^p(0,T;L^{\frac{4p}{3p-4}}(\Gamma))$ for $2 \le p < 4$ and choosing $p = 3$ yields that $\tilde{u}^N$ is bounded in $L^3(0,T;L^{12/5}(\Gamma))$.

The last last term in (4.62) is bounded by the same arguments, with $D_u(q(u^N, v^N))$ replaced by $D_v(q(u^N, v^N))$ and $\tilde{u}^N$ replaced by $\tilde{v}^N$.

Similarly, we find bounds on $\partial_t \tilde{\theta}^N$ and $\partial_t \tilde{\varphi}^N$ in $L^2(0,T;H^{-1}(\Gamma))$ from (4.38) and (4.35) respectively. These also imply a bound on $\partial_t \tilde{v}^N \in L^2(0,T;H^{-1}(\Gamma))$.

These bounds on the time derivatives allow us to deduce

$$\partial_t \tilde{u}^N \rightharpoonup \partial_t \tilde{u} \text{ in } L^2(0,T;\left(H^1(B)\right)'),\ \partial_t \tilde{\varphi}^N \rightharpoonup \partial_t \tilde{\varphi} \text{ in } L^2(0,T;H^{-1}(\Gamma)),$$
$$\text{and } \partial_t \tilde{v}^N \rightharpoonup \partial_t \tilde{v} \text{ in } L^2(0,T;H^{-1}(\Gamma)).$$

If we recall the proof of Theorem 4.2, we also see that in addition we can infer

$$\operatorname{tr} \tilde{u}^N \to \operatorname{tr} \tilde{u} \text{ in } L^2(0,T;L^2(\Gamma)),\ \tilde{\varphi}^N \to \tilde{\varphi} \text{ in } L^2(0,T;L^2(\Gamma)), \text{ and}$$
$$\tilde{v}^N \to \tilde{v} \text{ in } L^2(0,T;L^2(\Gamma)).$$

In all these cases, the convergence also holds pointwise almost everywhere.

Lemma 4.1 with $D_u(q(u^N, v^N))$ and $D_v(q(u^N, v^N))$ instead of $q(u^N, v^N)$ yields the weak convergences

$$D_u(q(u^N, v^N)) \rightharpoonup D_u q(u,v) \text{ in } L^2(0,T;L^2(\Gamma)) \text{ and}$$
$$D_v(q(u^N, v^N)) \rightharpoonup D_u q(u,v) \text{ in } L^2(0,T;L^2(\Gamma)).$$

Together with the foregoing results on the convergence of $\{\tilde{u}^N\}_{N\in\mathbb{N}}$ and $\{\tilde{v}^N\}_{N\in\mathbb{N}}$ this is sufficient to take the limit in the equations (4.34) and (4.38).

It remains to discuss the nonlinear term in (4.36). We first observe that $\varphi^N \to \varphi$ in $L^2(0,T;L^2(\Gamma))$ implies for almost all $x \in \Gamma$ the pointwise convergence

$$W''(\varphi^N(x)) \to W''(\varphi(x))$$

since W'' is continuous. Moreover, $W''(\varphi^N)$ is bounded in $L^2(0,T;L^2(\Gamma))$ as a consequence of (4.31). This yields (up to a subsequence) the weak convergence

$$W''(\varphi^N(x)) \rightharpoonup g \text{ in } L^2(0,T;L^2(\Gamma))$$

for some $g \in L^2(0,T;L^2(\Gamma))$. We repeat again that pointwise and weak limit must coincide if they both exist (see [DiB02, Proposition 9.1c]) and deduce

$$W''(\varphi^N(x)) \rightharpoonup W''(\varphi) \text{ in } L^2(0,T;L^2(\Gamma)).$$

This weak convergence together with the strong convergence $\tilde{\varphi}^N \to \tilde{\varphi}$ in $L^2(0,T;L^2(\Gamma))$ is sufficient to deduce

$$\int_0^T \int_\Gamma W''(\varphi^N)\tilde{\varphi}^N \omega \to \int_0^T \int_\Gamma W''(\varphi)\tilde{\varphi}\omega.$$

Hence we are able to take the limit in (4.36).

The remaining terms in the equations (4.34)–(4.38) are linear in $(\tilde{u}^N, \tilde{\varphi}^N, \tilde{v}^N, \tilde{\mu}^N, \tilde{\theta}^N)$ which implies that the limit functions $(\tilde{u}, \tilde{\varphi}, \tilde{v}, \tilde{\mu}, \tilde{\theta})$ are weak solutions to (4.56)–(4.61), first for all test functions ω and κ in $V_\Gamma^{N_0}$ and $V_B^{N_0}$ for some $N_0 \in \mathbb{N}$ respectively and by an analogue argument as at the end of the proof of Theorem 4.2 subsequently also for all test functions $\omega \in H^1(0,T;H^1(\Gamma))$ and $\kappa \in H^1(0,T;H^1(B))$. As such, $(\tilde{u}, \tilde{\varphi}, \tilde{v}, \tilde{\mu}, \tilde{\theta})$ are a weak solution to (4.56)–(4.61). $\square$

Third step: Higher regularity for the full system. We would like to apply elliptic regularity theory to equation (4.59). So far we have seen that $\varphi, \tilde{\varphi} \in L^\infty(0,T;H^1(\Gamma))$. As before, the Sobolev embedding theorem thus yields $\varphi, \tilde{\varphi} \in L^\infty(0,T;L^p(\Gamma))$ for all $1 \le p < \infty$. The term $W''(\varphi)\tilde{\varphi}$ on the right hand-side in (4.59) is an element of $L^2(0,T;L^2(\Gamma))$ because $|W''(\varphi)| \le C(1+|\varphi|^2)$ and Hölder's inequality thus implies

$$\left\|W''(\varphi)\tilde{\varphi}\right\|_{L^2(0,T;L^2(\Gamma))} \le C \left\|\varphi\right\|_{L^\infty(0,T;H^1(\Gamma))} \left\|\tilde{\varphi}\right\|_{L^\infty(0,T;H^1(\Gamma))}.$$

Hence the right hand-side in (4.59) is in $L^2(0,T;L^2(\Gamma))$ and as a first step we deduce

$$\tilde{\varphi} \in L^2(0,T;H^2(\Gamma)) \hookrightarrow L^2(0,T;W^{1,p}(\Gamma)) \text{ for all } 1 \le p < \infty$$

from elliptic theory. We can improve this result, since actually $W''(\varphi)\tilde{\varphi} \in L^2(0,T;H^1(\Gamma))$ by the following argument. The gradient of $W''(\varphi)\tilde{\varphi}$ can be estimated by

$$\begin{aligned}
&\int_0^T \int_\Gamma \left|\nabla_\Gamma \left(W''(\varphi)\tilde{\varphi}\right)\right|^2 \le \int_0^T \int_\Gamma \left|W''(\varphi)\nabla_\Gamma\tilde{\varphi}\right|^2 + \int_0^T \int_\Gamma \left|W'''(\varphi)\nabla_\Gamma\varphi\tilde{\varphi}\right|^2 \\
&\le \int_0^T \left(\int_\Gamma \left|W''(\varphi)\right|^4\right)^{1/2} \left(\int_\Gamma |\nabla_\Gamma\tilde{\varphi}|^4\right)^{1/2} + \int_0^T \left(\int_\Gamma \left|W'''(\varphi)\right|^8\right)^{1/4} \left(\int_\Gamma |\tilde{\varphi}|^8\right)^{1/4} \left(\int_\Gamma |\nabla_\Gamma\varphi|^4\right)^{1/2},
\end{aligned}$$

which implies

$$\begin{aligned}
\left\|\nabla_\Gamma \left(W''(\varphi)\tilde{\varphi}\right)\right\|_{L^2(0,T;L^2(\Gamma))} \le{}& \left\|W''(\varphi)\right\|_{L^\infty(0,T;L^4(\Gamma))} \left\|\nabla_\Gamma\tilde{\varphi}\right\|_{L^2(0,T;L^4(\Gamma))} \\
&+ \left\|W'''(\varphi)\right\|_{L^\infty(0,T;L^8(\Gamma))} \left\|\tilde{\varphi}\right\|_{L^\infty(0,T;L^8(\Gamma))} \left\|\nabla_\Gamma\varphi\right\|_{L^2(0,T;L^4(\Gamma))}.
\end{aligned}$$

The regularity of φ in (4.33) and of $\tilde{\varphi}$ above thus imply $W''(\varphi)\tilde{\varphi} \in L^2(0,T;H^1(\Gamma))$ and in turn we deduce from elliptic theory applied to (4.59)

$$\tilde{\varphi} \in L^2(0,T;H^3(\Gamma)).$$

Since

$$\partial_t u = \tilde{u} \text{ etc.}$$

in the sense of distributions as a direct consequence of the weak convergences in (4.52)– (4.55), this implies

$$\partial_t u \in L^\infty(0,T;L^2(B)) \cap L^2(0,T;H^1(B)), \tag{4.63}$$
$$\partial_t \varphi \in L^\infty(0,T;H^1(\Gamma)) \cap L^2(0,T;H^3(\Gamma)) \text{ and} \tag{4.64}$$
$$\partial_t \theta \in L^\infty(0,T;L^2(\Gamma)) \cap L^2(0,T;H^1(\Gamma)). \tag{4.65}$$

hence we can derive

$$\mu \in L^\infty(0,T;H^3(\Gamma)) \cap L^2(0,T;H^5(\Gamma))$$

because φ and μ are weak solutions to (2.4).

Recall that $u \in L^\infty(0,T;L^2(B)) \cap L^2(0,T;H^1(B))$ is a weak solution to

$$\begin{aligned}
\partial_t u &= D\Delta u && \text{in } B \times (0,T], \\
-D\nabla u \cdot \nu &= q(u,v) && \text{on } \Gamma \times (0,T],
\end{aligned}$$

where by the growth condition on $q(u,v)$ one can directly prove that $q(u,v) \in L^2(0,T;L^2(\Gamma))$ and from (4.63) we also have $\partial_t u \in L^2(0,T;L^2(B))$. Considering the more abstract elliptic problem

$$\begin{aligned} D\Delta u &= f \qquad \text{in } B \times (0,T], \\ -D\nabla u \cdot \nu &= g \qquad \text{on } \Gamma \times (0,T], \end{aligned}$$

we infer from Amann [Ama93, Remark 9.5 (a)] that this problem admits a solution $u \in H^1(B)$ for any

$$(f,g) \in H^{-1}(B) \times H^{-1/2}(\Gamma)$$

if and only if $\int_B f + \int_\Gamma g = 0$. We denote the corresponding continuous solution operator by

$$T : H^{-1}(B) \times H^{-1/2}(\Gamma) \to H^1(B).$$

On the other hand, it follows from the same reference or alternatively from [McL00, Theorem 4.18] that T is also continuous as an operator

$$T : L^2(B) \times H^{1/2}(\Gamma) \to H^2(B). \tag{4.66}$$

This allows us to consider the interpolation spaces (compare Remark 3.10)

$$\begin{aligned} H^{-1/2}(B) &= \left(H^{-1}(B), L^2(B)\right)_{1/2,2}, \\ L^2(\Gamma) &= \left(H^{-1/2}(\Gamma), H^{1/2}(\Gamma)\right)_{1/2,2}, \text{ and} \\ H^{3/2}(B) &= \left(H^1(B), H^2(B)\right)_{1/2,2} \end{aligned}$$

to deduce from the defintion of interpolation spaces 3.2 that T must also be continuous as an operator

$$T : H^{-1/2}(B) \times L^2(\Gamma) \to H^{3/2}(B).$$

Given that $q(u,v) \in L^2(0,T;L^2(\Gamma))$ and $\partial_t u \in L^2(0,T;L^2(B))$, we deduce that

$$u \in L^2(0,T;H^{3/2}(B)).$$

Together with (4.63) we infer $u \in H^1(0,T;H^{1/2}(\Gamma))$ and in particular

$$u \in L^\infty(0,T;H^{1/2}(\Gamma)) \hookrightarrow L^\infty(0,T;L^4(\Gamma))$$

because of the Sobolev embedding theorem. Using $v = \frac{\delta}{4}\theta + \frac{1}{2}\varphi$, (4.64), and (4.65) we derive the same property for v. Since $D_u q(u,v)$ and $D_v q(u,v)$ grow at most linearly by (4.27) and (4.28), we thus have

$$D_u q(u,v), D_v q(u,v) \in L^\infty(0,T;L^4(\Gamma)).$$

We use this information to derive that

$$\begin{aligned} \|D_u q(u,v)\nabla_\Gamma u\|^2_{L^2(0,T;L^{4/3}(\Gamma))} &= \int_0^T \left(\int_\Gamma |D_u q(u,v)|^{4/3} |\nabla_\Gamma u|^{4/3}\right)^{3/2} \\ &\le \int_0^T \left(\left(\int_\Gamma |D_u q(u,v)|^4\right)^{1/2} \left(\int_\Gamma |\nabla_\Gamma u|^2\right)\right) \\ &\le C \|D_u q(u,v)\|_{L^\infty(0,T;L^4(\Gamma))} \left(\int_0^T \int_\Gamma |\nabla_\Gamma u|^2\right) \end{aligned}$$

and

$$\|D_v q(u,v)\nabla_\Gamma v\|^2_{L^2(0,T;L^{4/3}(\Gamma))} \le C\,\|D_v q(u,v)\|_{L^\infty(0,T;L^4(\Gamma))}\left(\int_0^T\int_\Gamma |\nabla_\Gamma v|^2\right),$$

from which we obtain that

$$\nabla_\Gamma\left(q(u,v)\right) = D_u q(u,v)\nabla_\Gamma u + D_v q(u,v)\nabla_\Gamma v \in L^2(0,T;L^{4/3}(\Gamma)).$$

We recall that $q(u,v)\in L^2(0,T;L^2(\Gamma))$ and deduce

$$q(u,v)\in L^2(0,T;W^{1,4/3}(\Gamma)) \hookrightarrow L^2(0,T;H^{1/2}(\Gamma))$$

from the Sobolev embedding theorem. Thus the mapping properties in (4.66) actually yield

$$u \in L^2(0,T;H^2(B)). \tag{4.67}$$

We have already seen that $W'(\varphi)\in L^\infty(0,T;L^2(\Gamma))$ as well as $\theta,\mu\in L^\infty(0,T;L^2(\Gamma))$. As φ is a solution to (2.5), we thus deduce

$$\varphi \in L^\infty(0,T;H^2(\Gamma)). \tag{4.68}$$

Moreover, $\partial_t v\in L^\infty(0,T;L^2(\Gamma))\cap L^2(0,T;H^1(\Gamma))$ by (4.64), (4.65), and (2.7). Since θ solves (2.6) and in addition $q(u,v)\in L^2(0,T:L^2(\Gamma))$, we deduce

$$\theta\in L^2(0,T;H^2(\Gamma))$$

from elliptic regularity theory. Thus

$$v = \frac{\delta}{4}\theta + \frac{1}{2}\varphi \in L^2(0,T;H^2(\Gamma)).$$

By (4.67) we have $u\in L^2(0,T;H^{3/2}(\Gamma))$ and in particular $\nabla_\Gamma u\in L^2(0,T;L^4(\Gamma))$. We repeat the calculations from before to deduce

$$\|D_u q(u,v)\nabla_\Gamma u\|^2_{L^2(0,T;L^2(\Gamma))} \le C\,\|D_u q(u,v)\|_{L^\infty(0,T;L^4(\Gamma))}\left(\int_0^T\int_\Gamma |\nabla_\Gamma u|^4\right)^{1/2},$$

i.e. $D_u q(u,v)\nabla_\Gamma u\in L^2(0,T;L^2(\Gamma))$. Furthermore, $v\in L^2(0,T;H^2(\Gamma))$ yields $D_v q(u,v)\nabla_\Gamma v\in L^2(0,T;L^2(\Gamma))$ in a completely analogous manner.

As a direct consequence, we infer that in fact $q(u,v)\in L^2(0,T;H^1(\Gamma))$. Together with $\partial_t v\in L^\infty(0,T;L^2(\Gamma))\cap L^2(0,T;H^1(\Gamma))$ we turn again to elliptic regularity theory to deduce

$$\theta\in L^\infty(0,T;H^2(\Gamma))\cap L^2(0,T;H^3(\Gamma)).$$

We return to the regularity of φ in (4.68). $H^2(\Gamma)$ is a Banach algebra by Remark 3.15 and hence every polynomial in φ belongs to $L^\infty(0,T;H^2(\Gamma))$. In particular, this holds true for $W''(\varphi)$. Therefore, we can estimate

$$\int_0^T \left\|\nabla_\Gamma\left(W'(\varphi)\right)\right\|^2_{H^2(\Gamma)} = \int_0^T \left\|W''(\varphi)\nabla_\Gamma\varphi\right\|^2_{H^2(\Gamma)} \le \left\|W''(\varphi)\right\|_{L^\infty(0,T;H^2)}\int_0^T \|\nabla_\Gamma\varphi\|^2_{H^2(\Gamma)}$$

where we can use (4.33) to control the last term. Hence

$$W'(\varphi)\in L^2(0,T;H^3(\Gamma)),$$

and as a consequence

$$\varphi\in L^2(0,T;H^5(\Gamma))\cap L^\infty(0,T;H^2(\Gamma)),$$

which completes the proof of Theorem 4.3. □

Remark 4.7. The growth assumptions (4.27) is clearly satisfied in the non-equilibrium example

$$q(u,v) = c_1 u(1-v) - c_2 v$$

because $D_u q(u,v)$ is actually independent of u. If we modify q with suitable cut-off functions to fulfil the linear growth condition and consider $\tilde{q}$ as in Remark 2.8, we obtain $D_u\tilde{q}(u,v) = c_1 - c_1\eta'(u)v$ and $D_v\tilde{q}(u,v) = c_1\eta(u) - c_2$. Hence (4.27) and (4.28) hold if we choose the cut-off η in such a way that in addition to η its derivative η' is bounded as well.

Remark 4.8. It is a natural question to ask whether the growth assumption (4.27) can be weakened, for example by choosing different exponents in the Hölder inequality leading to (4.50). A cumbersome calculation analogue to (4.50) allows us to deduce for $\alpha > 2$ and $\tilde{\alpha}$ such that $1 = \frac{1}{\alpha} + \frac{2}{\tilde{\alpha}}$ the more general estimate

$$\begin{aligned}\int_\Gamma \left|D_u q(u^N, v^N)\right| \left|\tilde{u}^N\right|^2 &\leq \left\|D_u q(u^N, v^N)\right\|_{L^\alpha(\Gamma)} \left\|\tilde{u}^N\right\|^2_{L^{\tilde{\alpha}}(\Gamma)} \\ &\leq C_\gamma \left\|D_u q(u^N, v^N)\right\|^{\frac{2\alpha}{\alpha-2}}_{L^\alpha(\Gamma)} \left\|\tilde{u}^N\right\|^2_{L^2(B)} + \gamma \left\|\tilde{u}^N\right\|^2_{H^1(B)}.\end{aligned}$$

Since we are mostly interested in the dependence of $D_u q(u,v)$ on u, we simplify the discussion by assuming that $|D_u q(u,v)| \leq C|u|^l$ for some $l > 0$. Under this assumption, we have

$$\left\|D_u q(u^N, v^N)\right\|_{L^{\frac{2\alpha}{\alpha-2}}(0,T;L^\alpha(\Gamma))} \leq C \left\|u^N\right\|^l_{L^{\frac{2\alpha l}{\alpha-2}}(0,T;L^{\alpha l}(\Gamma))}.$$

For $u^N \in L^2(0,T;H^1(B)) \cap L^\infty(0,T;L^2(B))$ we obtain as in the proof of Lemma 4.4 that $u^N \in L^p(0,T;L^{\frac{4p}{3p-4}}(\Gamma))$ for $2 \leq p \leq 4$. In the view of the foregoing estimate, this corresponds to

$$\alpha l = \frac{4p}{3p-4}, \text{ and } p = \frac{2\alpha l}{\alpha - 2}.$$

Observe that for $l = 1$ there are no solutions to this system of equations such that $2 \leq p \leq 4$ and $\alpha > 2$, i.e. some growth rate lower then $l = 1$ is necessary.

We plug the second equation into the first to arrive at

$$\alpha l = \frac{8\alpha l}{-4\alpha + 8 + 6\alpha l},$$

provided that $-4\alpha + 8 + 6\alpha l \neq 0$. If this is the case, we divide the equation by $\alpha l > 0$ and deduce $l = \frac{2}{3}$. Otherwise, $-4\alpha + 8 + 6\alpha l = 0$ leads to $l = \frac{4\alpha-8}{6\alpha}$. For $\alpha > 2$, this is always smaller then $\frac{2}{3}$. This observation justifies the assumption (4.27).

4.2 Convergence to the Reduced Model as $D \to \infty$

In Section 2.4 we formally derived the reduced model (2.28)–(2.32) based on the observation that Corollary 2.6 implied that we can expect u to be spatially constant in the limit $D \to \infty$. After we proved the necessary estimate (2.27) rigorously in Theorem 4.2, we are now in the position to establish the connection between the full model (2.2)–(2.7) and the reduced model (2.28)–(2.32) rigorously.

Proposition 4.9. *Let* $\{D_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ *be a sequence with* $\lim_{n\to\infty} D_n = \infty$ *and denote by* $(u^{D_n}, \varphi^{D_n}, \mu^{D_n}, \theta^{D_n}, v^{D_n})$ *the weak solution from Theorem 4.2 with* $D = D_n$ *and initial data independent of* n*. Then there exists a subsequence (again denoted by* $\{D_n\}_{n\in\mathbb{N}}$*) such that*

$$\begin{aligned}
u^{D_n} &\rightharpoonup u \text{ in } L^2(0,T;H^1(B)) \cap H^1(0,T;H^{-1}(B)) \text{ with } u(t) \in \mathbb{R}\,\forall t \in (0,T),\\
\varphi^{D_n} &\rightharpoonup \varphi \text{ in } L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma)),\\
\mu^{D_n} &\rightharpoonup \mu \text{ in } L^2(0,T;H^1(\Gamma)),\\
\theta^{D_n} &\rightharpoonup \theta \text{ in } L^2(0,T;H^1(\Gamma)),\\
v &\rightharpoonup v \text{ in } L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma)),
\end{aligned}$$

and such that the limit functions are weak solution to the reduced problem (2.28)–(2.32)*, i.e. they fulfil for all* $\xi \in L^2(0,T;H^1(B))$ *and* $\eta \in L^2(0,T;H^1(\Gamma))$ *the equations*

$$\begin{aligned}
\int_0^T \langle \partial_t u, \xi\rangle_{(H^1(B))',H^1(B)} &= -\frac{1}{|B|}\int_0^T\int_\Gamma q(u,v)\xi,\\
\int_0^T \langle \partial_t \varphi, \eta\rangle_{H^{-1}(\Gamma),H^1(\Gamma)} &= -\int_0^T\int_\Gamma \nabla_\Gamma \mu\cdot\nabla_\Gamma\eta,\\
\int_0^T\int_\Gamma \mu\eta &= -\int_0^T\int_\Gamma \varepsilon\nabla_\Gamma\varphi\cdot\nabla_\Gamma\eta + \frac{1}{\varepsilon}\int_0^T\int_\Gamma W'(\varphi)\eta - \frac{1}{\delta}\int_0^T\int_\Gamma (2v-1-\varphi)\,\eta,\\
\int_0^T \langle \partial_t v, \eta\rangle_{H^{-1}(\Gamma),H^1(\Gamma)} &= -\int_0^T\int_\Gamma \nabla_\Gamma\theta\cdot\nabla_\Gamma\eta + \int_0^T\int_\Gamma q(u,v)\eta,\\
\int_0^T\int_\Gamma \theta\eta &= \frac{2}{\delta}\int_\Gamma (2v-1-\varphi)\,\eta.
\end{aligned}$$

The initial values are attained in $L^2(B)$ *and* $L^2(\Gamma)$ *respectively.*

Proof. According to Theorem 4.2, the solution $(u^{D_n}, \varphi^{D_n}, \mu^{D_n}, \theta^{D_n}, v^{D_n})$ fulfils

$$\begin{aligned}
\sup_{0\le t\le T}&\left\{\frac{1}{2}\int_B \left|u^{D_n}\right|^2 + \frac{\varepsilon}{2}\int_\Gamma \left|\nabla_\Gamma\varphi^{D_n}\right|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi^{D_n}) + \frac{\delta}{8}\int_\Gamma \left|\theta^{D_n}\right|^2\right\}\\
&+ \frac{D_n}{2}\int_0^T\int_B \left|\nabla u^{D_n}\right|^2 + \int_0^T\int_\Gamma \left|\nabla_\Gamma\mu^{D_n}\right|^2 + \int_0^T\int_\Gamma \left|\nabla_\Gamma\theta^{D_n}\right|^2 \le C(T). \qquad (4.69)
\end{aligned}$$

We exploit (2.2) to deduce that for all $\tau \in H^1(B)$

$$\begin{aligned}
\left|\langle \partial_t u^{D_n}, \tau\rangle\right| = \left|\int_B \partial_t u^{D_n}\tau\right| &\le D\left|\int_B \nabla u^{D_n}\cdot\nabla\tau\right| + \left|\int_\Gamma q(u^{D_n}, v^{D_n})\tau\right|\\
&\le D\left\|u^{D_n}\right\|_{H^1(B)}\|\tau\|_{H^1(B)} + \left\|q(u^{D_n}, v^{D_n})\right\|_{L^2(\Gamma)}\|\tau\|_{L^2(\Gamma)}.
\end{aligned}$$

Hence we obtain

$$\left\|\partial_t u^{D_n}\right\|_{L^2(0,T;(H^1(B))')} \le C(T)$$

from (4.69). Moreover, (2.4) and (2.6) imply

$$\left\|\partial_t\varphi^{D_n}\right\|_{L^2(0,T;H^{-1}(\Gamma))} \le C(T) \text{ and } \left\|\partial_t v^{D_n}\right\|_{L^2(0,t;H^{-1}(\Gamma))} \le C(T)$$

by similar arguments. Thus the time derivatives fulfil

$$\begin{aligned}
\partial_t u^{D_n} &\rightharpoonup \partial_t u \text{ in } L^2(0,T;\big(H^1(B)\big)'),\\
\partial_t \varphi^{D_n} &\rightharpoonup \partial_t \varphi \text{ in } L^2(0,T;H^{-1}(\Gamma)),\\
\partial_t v^{D_n} &\rightharpoonup \partial_t v \text{ in } L^2(0,T;H^{-1}(\Gamma)).
\end{aligned}$$

The estimate (4.69) furthermore yields the existence of subsequences (again denoted by D_n) such that

$$\begin{aligned}
u^{D_n} &\rightharpoonup u \text{ in } L^2(0,T;H^1(B)),\\
u^{D_n} &\to u \text{ in } L^2(0,T;H^s(B)), 0 < s < 1,\\
\operatorname{tr}(u^{D_n}) \to \operatorname{tr}(u) \text{ in } L^2(0,T;L^2(\Gamma)) &\text{ and } \operatorname{tr}(u^{D_n})(x) \to \operatorname{tr}(u)(x) \text{ a.e. in } \Gamma_T,\\
\varphi^{D_n} \rightharpoonup \varphi \text{ in } L^2(0,T;H^1(\Gamma)) &\text{ and } \varphi^{D_n} \to \varphi \text{ in } L^2(0,T;L^2(\Gamma)),\\
\mu^{D_n} &\rightharpoonup \mu \text{ in } L^2(0,T;H^1(\Gamma)),\\
\theta^{D_n} &\rightharpoonup \theta \text{ in } L^2(0,T;H^1(\Gamma)),\\
v^{D_n} \rightharpoonup v \text{ in } L^2(0,T;H^1(\Gamma)) &\text{ and } v^{D_n} \to v \text{ in } L^2(0,T;L^2(\Gamma)),\\
v^{D_n}(x) &\to v(x) \text{ a.e. in } \Gamma_T.
\end{aligned}$$

The strong convergences $v^{D_n} \to v$ and $\varphi^{D_n} \to \varphi$ in $L^2(0,T;L^2(\Gamma))$ here are a consequence of the Aubin-Lions theorem. We remark that these arguments are completely analogue to the proof of Theorem 4.2 and we thus omit some details.

In particular, (4.69) implies

$$\int_0^T \int_B \left|\nabla u^{D_n}\right|^2 \leq \frac{C(T)}{D_n} \to 0 \text{ as } n \to \infty.$$

Thus $\nabla u \equiv 0$ and the limit function u is constant in the space variables.

It remains to discuss the limit process within the equations. Again, we refer to the proof of Theorem 4.2 for the details since the arguments in both cases are completely analogue. Lemma 4.1 yields

$$q(u^{D_n}, v^{D_n}) \rightharpoonup q(u,v) \text{ in } L^2(0,T;L^2(\Gamma)).$$

As before, we take the use this information together with $\partial_t u^{D_n} \rightharpoonup \partial_t u$ in $L^2(0,T;\big(H^1(B)\big)')$ to take the limit in (4.1). The second term vanishes in the limit since $\nabla u \equiv 0$. Estimate (4.69) implies that $W'(\varphi^{D_n})$ converges (a) pointwise almost everywhere to $W'(\varphi)$ and (b) weakly in $L^{4/3}(\Gamma_T)$. Hence

$$W'(\varphi^{D_n}) \rightharpoonup W'(\varphi) \text{ in } L^{4/3}(\Gamma_T)$$

and we can take the limit in (4.2)–(4.5). $\square$

Chapter 5

Longtime Existence and Stationary Solutions for the Reduced Model

5.1 Stationary Solutions

The goal of this section is to prove the existence of stationary solutions to the reduced model. We work with the reformulation (2.36)–(2.44) from Section 2.5 and recall that $\varphi_\Gamma, v_\Gamma, \mu_\Gamma$, and θ_Γ denote the mean value free functions $\varphi_\Gamma := \varphi - \frac{1}{|\Gamma|}\int_\Gamma \varphi$ and so on. Stationary solutions to these equations need to fulfil

$$0 = \Delta_\Gamma \mu_\Gamma \quad \text{on } \Gamma, \tag{5.1}$$

$$\mu_\Gamma = -\varepsilon\Delta_\Gamma\varphi_\Gamma + \varepsilon^{-1}P_\Gamma W'(\varphi) - \frac{\theta_\Gamma}{2} \quad \text{on } \Gamma, \tag{5.2}$$

$$0 = \Delta_\Gamma\theta_\Gamma + P_\Gamma q(u,v) \quad \text{on } \Gamma, \tag{5.3}$$

$$\theta_\Gamma = \frac{2}{\delta}(2v_\Gamma - \varphi_\Gamma) \quad \text{on } \Gamma \tag{5.4}$$

together with the equations

$$0 = \int_\Gamma q(u,v) \tag{5.5}$$

$$\int_\Gamma \varphi = m \tag{5.6}$$

$$\int_\Gamma v = M - \int_B u \tag{5.7}$$

$$\int_\Gamma \mu = \int_\Gamma \left(\varepsilon^{-1}W'(\varphi) + \frac{\theta}{2}\right) \tag{5.8}$$

$$\int_\Gamma \theta = \frac{2}{\delta}\int_\Gamma [2v - 1 - \varphi] \tag{5.9}$$

for given mass constraints $m, M \in \mathbb{R}$.

Condition 5.1. We assume that there exists an continuous operator $S : H^1_{(0)}(\Gamma) \to \mathbb{R}^2$ such that for all $\tilde{v} \in H^1_{(0)}(\Gamma)$ and for any given $M \in \mathbb{R}$ the pair $(\overline{u}, \overline{v}) := S(\tilde{v}) \in \mathbb{R}^2$ solves

$$\int_\Gamma q(\overline{u}, \tilde{v} + \overline{v}) = 0,$$
$$\int_B \overline{u} + \int_\Gamma \overline{v} = M.$$

We also write $S_B(\tilde{v}) = \overline{u}$ and $S_\Gamma(\tilde{v}) = \overline{v}$.

Remark 5.2. Condition 5.1 is there to ascertain that the mean values $\frac{1}{|B|}\int_B u$ and $\frac{1}{|\Gamma|}\int_\Gamma v$ are determined by the two equations (5.5) and (5.7). Remark 2.8 shows that this condition is satisfied for the prime example (2.23) in the non-equilibrium case

$$q(u, v) = c_1 u(1 - v) - c_2 v,$$

since in this case $\int_\Gamma q(u, v)$ does not depend on v as can be seen from (2.35).

Theorem 5.3. *Let $m, M \in \mathbb{R}$ be given. Assume that $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has sublinear growth, i.e. assume that there exists $\alpha > 1$ such that*

$$|q(u, v)| \leq C\left(1 + |u|^{1/\alpha} + |v|^{1/\alpha}\right). \tag{5.10}$$

Moreover, assume that q fulfils Condition 5.1. Then there exist $u \in \mathbb{R}$ and functions

$$(\varphi, v, \mu, \theta) \in H^1(\Gamma) \times H^1(\Gamma) \times H^1(\Gamma) \times H^1(\Gamma)$$

which are weak solutions of (5.1)–(5.9).

Proof of Theorem 5.3. W.l.o.g we can assume that the mean value of φ vanishes, i.e. $\frac{1}{|\Gamma|}\int_\Gamma \varphi = m = 0$. This is due to the fact that we can always consider $\overline{\varphi} = \varphi - m$ and work with the translated double-well potential $\overline{W}(s) = W(s + m)$.

We first consider the equations (5.1)–(5.4) for the mean value free functions $\varphi_\Gamma, v_\Gamma, \theta_\Gamma, \mu_\Gamma$. Note that these equations do not depend on the mean value $\int_\Gamma \mu$.

In particular, equation (5.1) implies that μ_Γ is constant. Since $\int_\Gamma \mu_\Gamma = 0$, we thus directly deduce $\mu_\Gamma = 0$. As such, equations (5.1)–(5.4) reduce to

$$\begin{aligned} 0 &= -\varepsilon \Delta_\Gamma \varphi_\Gamma + \varepsilon^{-1} P_\Gamma W'(\varphi) - \frac{\theta_\Gamma}{2} && \text{on } \Gamma, \\ 0 &= \Delta_\Gamma \theta_\Gamma + P_\Gamma q(u, v) && \text{on } \Gamma, \\ \theta_\Gamma &= \frac{2}{\delta}(2 v_\Gamma - \varphi_\Gamma) && \text{on } \Gamma. \end{aligned}$$

To begin with, recall that $W'(\varphi) = 4\varphi^3 - 4\varphi$ and that the projection P_Γ is linear.

Let Z denote the space

$$Z := H^1_{(0)}(\Gamma) \times H^1_{(0)}(\Gamma) \times H^1_{(0)}(\Gamma)$$

and define for $\tau \geq 0$ by T_τ the solution operator which maps a given right hand side $(\tilde{\varphi}, \tilde{v}, \tilde{\theta}) \in Z$ onto the solution to the problem

$$0 = -\varepsilon\Delta_\Gamma\varphi_\Gamma + 4\varepsilon^{-1}P_\Gamma\big(\big(\varphi_\Gamma^3 - \tau\tilde{\varphi}\big)\big) - \frac{\theta_\Gamma}{2} \qquad \text{on } \Gamma, \tag{5.11}$$

$$0 = \Delta_\Gamma\theta_\Gamma + \tau P_\Gamma q(S_B(\tilde{v}), \tilde{v} + S_\Gamma(\tilde{v})) \qquad \text{on } \Gamma, \tag{5.12}$$

$$\theta_\Gamma = \frac{2}{\delta}(2v_\Gamma - \varphi_\Gamma) \qquad \text{on } \Gamma, \tag{5.13}$$

where S_B and S_Γ are the operators provided by Condition 5.1. Note that for $\tau = 0$, the operator $T_0 : Z \to Z$ maps every element of Z onto the solution to

$$\begin{aligned} 0 &= -\varepsilon\Delta_\Gamma\varphi_\Gamma + 4\varepsilon^{-1}P_\Gamma(\varphi_\Gamma^3) - \frac{\theta_\Gamma}{2} && \text{on } \Gamma,\\ 0 &= \Delta_\Gamma\theta_\Gamma && \text{on } \Gamma,\\ \theta_\Gamma &= \frac{2}{\delta}(2v_\Gamma - \varphi_\Gamma) && \text{on } \Gamma, \end{aligned}$$

i.e T_0 is constant.

Lemma 5.4. The operator $T_\tau : Z \to Z$ is well defined and compact.

Proof of Lemma 5.4. Since q has sublinear growth by assumption (5.10), $\tilde{v} \in H^1_{(0)}(\Gamma)$ is given, and S_B and S_Γ are continuous, we see that $\tau P_\Gamma q(S_B(\tilde{v}), \tilde{v} + S_\Gamma(\tilde{v})) \in L^2(\Gamma)$. Equation (5.12) has therefore a unique solution $\theta_\Gamma \in H^2_{(0)}(\Gamma)$.

Let $V(s) := s^4$ be the convex part of W. We now define $\mathcal{G} : L^2_{(0)}(\Gamma) \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{G}(\varphi_\Gamma) := \begin{cases} \int_\Gamma \frac{\varepsilon}{2}|\nabla_\Gamma\varphi_\Gamma|^2 + \frac{1}{\varepsilon}V(\varphi_\Gamma), & \text{if } \varphi_\Gamma \in H^1_{(0)}(\Gamma),\\ +\infty & \text{else.} \end{cases}$$

Then $\mathcal{G}$ is a proper, convex, and lower semi-continuous functional by Fatou's lemma. By [Bré73, Example 2.3.4], its L^2–gradient $\mathrm{A} : L^2_{(0)}(\Gamma) \supset D(\mathrm{A}) \to L^2_{(0)}(\Gamma)$ is therefore a maximal monotone operator, given by

$$\mathrm{A} = \left(-\varepsilon\Delta_\Gamma\varphi_\Gamma + \frac{1}{\varepsilon}P_\Gamma V'(\varphi_\Gamma)\right).$$

Its domain $D(\mathrm{A})$ is $D(\mathrm{A}) = H^2_{(0)}(\Gamma)$. Moreover, for all $\varphi_\Gamma \in D(\mathrm{A})$

$$\lim_{\|\varphi_\Gamma\|_{L^2_{(0)}(\Gamma)}\to\infty} \frac{\mathcal{G}(\varphi_\Gamma)}{\|\varphi_\Gamma\|_{L^2_{(0)}(\Gamma)}} \geq \lim_{\|\varphi_\Gamma\|_{L^2_{(0)}(\Gamma)}\to\infty} C\frac{\|\nabla_\Gamma\varphi_\Gamma\|^2}{\|\varphi_\Gamma\|_{H^1_{(0)}(\Gamma)}} = +\infty$$

and by Proposition 2.14 in [Bré73] we find that for every $f \in L^2_{(0)}(\Gamma)$ there exists a $\varphi_\Gamma \in D(\mathrm{A})$ which solves

$$A\varphi_\Gamma = f.$$

The solution φ_Γ is unique, because A is in fact strictly monotone. Indeed, already the L^2–gradient of $\int_\Gamma \frac{\varepsilon}{2}|\nabla_\Gamma\varphi_\Gamma|^2$ is strictly monotone and $\int_\Gamma V(\varphi_\Gamma)$ is convex itself. Choosing $f = \left(\tau\tilde{\varphi} + \frac{\theta_\Gamma}{2}\right)$ we have $f \in L^2_{(0)}(\Gamma)$ since $\theta_\Gamma \in H^2_{(0)}(\Gamma)$ and $\tilde{\varphi_\Gamma} \in H^1_{(0)}(\Gamma)$. Consequently, there exists a unique $\varphi_\Gamma \in D(\mathrm{A}) \subset H^2_{(0)}(\Gamma)$ which solves

$$A\varphi_\Gamma = \left(-\varepsilon\Delta_\Gamma\varphi_\Gamma + \frac{1}{\varepsilon}P_\Gamma V'(\varphi_\Gamma)\right) = \left(\tau\tilde{\varphi} + \frac{\theta_\Gamma}{2}\right),$$

i.e. equation (5.11).

To conclude the proof, we note that

$$H^2_{(0)}(\Gamma) \times H^2_{(0)}(\Gamma) \times H^2_{(0)}(\Gamma)$$

embeds compactly into Z, hence $T : Z \to Z$ is indeed compact. □

The proof of Theorem 5.3 is now based on a fixed point argument for T_1. By the Leray-Schauder theorem, we have a solution for the fixed point equation

$$T_1 \begin{pmatrix} \varphi_\Gamma \\ v_\Gamma \\ \theta_\Gamma \end{pmatrix} = \begin{pmatrix} \varphi_\Gamma \\ v_\Gamma \\ \theta_\Gamma \end{pmatrix}$$

if we can prove uniform a priori estimates for solutions to

$$T_\tau \begin{pmatrix} \varphi_\Gamma \\ v_\Gamma \\ \theta_\Gamma \end{pmatrix} = \begin{pmatrix} \varphi_\Gamma \\ v_\Gamma \\ \theta_\Gamma \end{pmatrix}, \tag{5.14}$$

where $\tau \in (0,1)$.

Lemma 5.5. Let $\tau \in (0,1)$ and let $(\varphi_\Gamma, v_\Gamma, \theta_\Gamma)$ be a solution to (5.14). Then

$$\int_\Gamma |\nabla_\Gamma \varphi_\Gamma|^2 + \int_\Gamma \varphi_\Gamma^4 + \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 \le C(\varepsilon, \Gamma, u) \tag{5.15}$$

and

$$\int_\Gamma |\nabla_\Gamma v_\Gamma|^2 \le C(\varepsilon, \Gamma, u, \delta) \tag{5.16}$$

Both estimates are uniform in τ.

Proof of Lemma 5.5. We multiply equation (5.11) by φ_Γ and equation (5.12) by θ_Γ. Taking the sum of both equations and integrating over Γ yields

$$\begin{aligned}\int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \varepsilon \int_\Gamma |\nabla_\Gamma \varphi_\Gamma|^2 + \frac{4}{\varepsilon}\int_\Gamma \varphi_\Gamma^4 \\ = -\frac{1}{2}\int_\Gamma \theta_\Gamma \varphi_\Gamma + \tau \int_\Gamma \varphi_\Gamma^2 + \tau \int_\Gamma q(u, v_\Gamma)\theta_\Gamma.\end{aligned}$$

We use that $\tau \in (0,1)$ and Young's inequality for $\eta, \gamma > 0$ to deduce

$$\begin{aligned}\int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \varepsilon \int_\Gamma |\nabla_\Gamma \varphi_\Gamma|^2 + \frac{4}{\varepsilon}\int_\Gamma \varphi_\Gamma^4 \\ \le (C(\eta) + 1)\int_\Gamma \varphi_\Gamma^2 + \eta \int_\Gamma \theta_\Gamma^2 + \left|\int_\Gamma q(u, v_\Gamma)\theta_\Gamma\right| \\ \le (C(\eta) + 1)\gamma \int_\Gamma \varphi_\Gamma^4 + \eta C \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 \\ + C(\eta, \gamma, \Gamma) + \left|\int_\Gamma q(u, v_\Gamma)\theta_\Gamma\right|.\end{aligned}$$

For η sufficiently small, this inequality implies

$$\int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \varepsilon \int_\Gamma |\nabla_\Gamma \varphi_\Gamma|^2 + \frac{4}{\varepsilon} \int_\Gamma \varphi_\Gamma^4 \\ \leq (C(\eta) + 1)\,\gamma \int_\Gamma \varphi_\Gamma^4 + C(\eta, \gamma, \Gamma) + \left| \int_\Gamma q(u, v_\Gamma)\theta_\Gamma \right|.$$

Subsequently, we choose γ sufficiently small to infer

$$\frac{1}{2} \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \varepsilon \int_\Gamma |\nabla_\Gamma \varphi_\Gamma|^2 + \frac{2}{\varepsilon} \int_\Gamma \varphi_\Gamma^4 \leq C(\eta, \gamma, \Gamma) + \left| \int_\Gamma q(u, v_\Gamma)\theta_\Gamma \right|.$$

By the assumptions (5.10) on q we can estimate the right hand side in the above equation by

$$\left| \int_\Gamma q(u, v_\Gamma)\theta_\Gamma \right| \leq C \int_\Gamma |\theta_\Gamma| \left(1 + |u|^{1/\alpha} + |v_\Gamma|^{1/\alpha} \right).$$

From Young's inequality we deduce

$$\int_\Gamma |\theta_\Gamma|\,|v_\Gamma|^{1/\alpha} \leq C \int_\Gamma |\theta_\Gamma|^{\frac{\alpha+1}{\alpha}} + \int_\Gamma |v_\Gamma|^{\frac{\alpha+1}{\alpha}} \leq C(\rho) + \rho \left(\int_\Gamma |\theta_\Gamma|^2 + \int_\Gamma |v_\Gamma|^2 \right)$$

since $\frac{2\alpha}{\alpha+1} > 1 \Leftrightarrow \alpha > 1$ and using equation (5.13) we obtain

$$\int_\Gamma |v_\Gamma|^2 \leq \frac{1}{2} \left(\int_\Gamma \left| \frac{\delta}{2} \theta_\Gamma \right|^2 + \int_\Gamma |\varphi_\Gamma|^2 \right).$$

Since $u \in \mathbb{R}$ is a given constant, the estimates for the remaining terms are straightforward and Poincaré's inequality yields

$$\int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \varepsilon \int_\Gamma |\nabla_\Gamma \varphi_\Gamma|^2 + \frac{4}{\varepsilon} \int_\Gamma \varphi_\Gamma^4 \\ \leq C(\eta, \gamma, \Gamma, \rho) + \rho C \left(\int_\Gamma \left| \frac{\delta}{2} \nabla_\Gamma \theta_\Gamma \right|^2 + \int_\Gamma |\nabla_\Gamma \varphi_\Gamma|^2 \right).$$

Choosing ρ sufficiently small, we deduce the estimate (5.15). Estimate (5.16) now follows directly from equation (5.13). □

Based on these two lemmas, we now proceed with the proof of Theorem 5.3.

By Lemma 5.4, $T_\tau : Z \to Z$ is a compact homotopy between the constant map $T_0 : Z \to Z$ and $T_1 : Z \to Z$. By Lemma 5.5 and the Poincaré inequality we have uniform a priori estimates in Z on all solutions to (5.14). The Leray-Schauder principle [Zei86, Theorem 6.A] (or the Leray-Schauder mapping degree theory, [Zei86, Chapter 13]) hence guarantees the existence of a fixed point of T_1, thus proving the theorem. □

5.2 Longtime Existence

In this section we prove that under suitable assumptions the surface energy $\mathcal{F}(v, \varphi)$ for the reduced system admits a bound which is uniform in time.

Condition 5.6. We assume that q grows at most linearly, i.e. there exists $\alpha > 1$ such that $|q(u, v)| \leq C(1 + |u|^{1/\alpha} + |v|^{1/\alpha})$ and that $u \in L^\infty(0, \infty)$.

Remark 5.7. 1. For all choices for the exchange term q, u is given as the the solution to the ordinary differential equation

$$\frac{d}{dt}\int_B u(t)\ dx = -\int_\Gamma q(u,v)$$

Therefore, Condition 5.6 is fulfiled if the solution to this equation exists for all times and stays bounded as $t \to \infty$. As we have already discussed in Remark 2.8(1) and (2), this is in particular the case for the prime example (2.23) in the non-equilibrium case with suitable initial values, namely

$$q(u,v) = c_1 u(1-v) - c_2 v.$$

In this case, the function u is given as the solution to the ordinary differential equation

$$\frac{d}{dt}\int_B u(t)\ dx = -\frac{c_1\,|\Gamma|}{|B|}\int_B u(t)\ dx + \left(\frac{c_1}{|B|}\int_B u(t)\ dx + c_2\right)\left(M - \int_B u\ dx\right)$$

as we discussed in Remark 2.8(1) and (2). The solution to this equation is bounded for all times for initial values in the interval $\left[0, M\,|B|^{-1}\right]$.

2. Remark 2.8(3) also shows that we can modify q as in (2.23) in such a way that the growth Condition 5.6 is fulfiled.

Lemma 5.8. Assume that Condition 5.6 holds. Then there exist constants $C, c > 0$ which do not depend on t such that

$$\frac{d}{dt}\mathcal{F}(v,\varphi) \le C - c\mathcal{F}(v,\varphi).$$

Proof. We calculate

$$\frac{d}{dt}\mathcal{F}(v,\varphi) = -\int_\Gamma |\nabla_\Gamma \mu_\Gamma|^2 - \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \int_\Gamma \theta q(u,v). \tag{5.17}$$

The last term can be estimated by

$$\begin{aligned}\left|\int_\Gamma \theta q(u,v)\right| &\le C\left(\int_\Gamma |\theta|\left|1 + |u|^{1/\alpha} + |v|^{1/\alpha}\right|\right)\\ &\le C\left(\int_\Gamma |\theta| + \int_\Gamma |\theta|\,|u|^{1/\alpha} + \int_\Gamma |\theta|\,|v|^{1/\alpha}\right).\end{aligned} \tag{5.18}$$

As before we find by Young's inequality

$$\int_\Gamma |\theta|\,|v|^{1/\alpha} \le C\left(\int_\Gamma |\theta|^{\frac{\alpha+1}{\alpha}} + \int_\Gamma |v|^{\frac{\alpha+1}{\alpha}}\right).$$

We note that $\frac{\alpha+1}{\alpha} > 1$ and hence conclude by Jensen's inequality

$$\begin{aligned}\int_\Gamma |\theta|\,|v|^{1/\alpha} &\le C\int_\Gamma |\theta|^{\frac{\alpha+1}{\alpha}} + C\int_\Gamma \left|\frac{\delta}{4}\theta + \frac{\varphi+1}{2}\right|^{\frac{\alpha+1}{\alpha}}\\ &\le \int_\Gamma |\theta|^{\frac{\alpha+1}{\alpha}} + C(\alpha)\int_\Gamma \left|\frac{\delta}{4}\theta\right|^{\frac{\alpha+1}{\alpha}} + C(\alpha)\int_\Gamma \left|\frac{\varphi+1}{2}\right|^{\frac{\alpha+1}{\alpha}}\end{aligned}$$

where we have also used that $v = \frac{\delta}{4}\theta + \frac{\varphi+1}{2}$. Since $|\Gamma| < \infty$, Hölder's inequality yields

$$\int_\Gamma |\theta|\,|v|^{1/\alpha} \leq C(\delta)\left(\int_\Gamma |\theta|^2\right)^{\frac{\alpha+1}{2\alpha}} + C\left(\int_\Gamma |\varphi+1|^2\right)^{\frac{\alpha+1}{2\alpha}}$$

If we take into account that $u(t) \in \mathbb{R}$ is uniformly bounded in t by Condition 5.6 and use Hölder's inequality to estimate the remaining terms in (5.18), we arrive at

$$\begin{aligned}\left|\int_\Gamma \theta q(u,v)\right| &\leq C_1\left(\int_\Gamma |\theta|^2\right)^{1/2} + C(\delta)\left(\int_\Gamma |\theta|^2\right)^{\frac{\alpha+1}{2\alpha}} + C_2\left(\int_\Gamma |\varphi+1|^2\right)^{\frac{\alpha+1}{2\alpha}} + C_3 \\ &\leq C(\delta)\left(\left(\int_\Gamma |\theta|^2\right)^{1/2} + \left(\int_\Gamma |\theta|^2\right)^{\frac{\alpha+1}{2\alpha}} + \left(\int_\Gamma |\varphi+1|^2\right)^{\frac{\alpha+1}{2\alpha}} + 1\right).\end{aligned}$$

We define $\beta := \max\{\frac{1}{2}, \frac{\alpha+1}{2\alpha}\} < 1$ and again using $|\Gamma| < \infty$ and Hölder's inequality arrive at

$$\left|\int_\Gamma \theta q(u,v)\right| \leq C(\delta)\left(\left(\int_\Gamma |\theta|^2\right)^{\beta} + \left(\int_\Gamma |\varphi+1|^2\right)^{\beta} + 1\right)$$

which implies

$$\left|\int_\Gamma \theta q(u,v)\right| \leq C(\delta)\mathcal{F}(v,\varphi)^\beta + C \tag{5.19}$$

since $\beta < 1$. If we multiply equation (2.37) by $\varphi_\Gamma = \varphi - ⨍_\Gamma \varphi$ and integrate over Γ we obtain

$$\int_\Gamma \mu\varphi_\Gamma + \frac{1}{2}\int_\Gamma \theta\varphi_\Gamma = \varepsilon\int_\Gamma |\nabla_\Gamma\varphi_\Gamma|^2 + \frac{1}{\varepsilon}\int_\Gamma W'(\varphi)\varphi_\Gamma.$$

The left hand side can be estimated by

$$\int_\Gamma \mu\varphi_\Gamma + \frac{1}{2}\int_\Gamma \theta\varphi_\Gamma \leq \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma\varphi_\Gamma|^2 + C\int_\Gamma |\nabla_\Gamma\mu|^2 + C\int_\Gamma |\nabla_\Gamma\theta|^2.$$

The double-well potential W fulfils $W'(s)s \geq c_0 W(s) - c_1$ for $c_0, c_1 > 0$. Thus the right hand side above can be estimated from below by

$$\varepsilon\int_\Gamma |\nabla_\Gamma\varphi_\Gamma|^2 + \frac{1}{\varepsilon}\int_\Gamma W'(\varphi)\varphi_\Gamma \geq \left(\int_\Gamma \varepsilon|\nabla_\Gamma\varphi_\Gamma|^2 + \frac{c_0}{\varepsilon}\int_\Gamma W(\varphi)\right) - \tilde{c}.$$

Both estimates imply

$$-\int_\Gamma |\nabla_\Gamma\mu|^2 - \int_\Gamma |\nabla_\Gamma\theta|^2 \leq -C\left(\int_\Gamma \varepsilon|\nabla_\Gamma\varphi_\Gamma|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi)\right) + \tilde{c}. \tag{5.20}$$

Next we observe that

$$\left|\int_\Gamma \theta\,\right| \leq |\Gamma|^{1/2}\left(\int_\Gamma |\theta|^2\right)^{1/2} \leq |\Gamma| + \int_\Gamma |\theta|^2 \leq \frac{2}{\delta}\mathcal{F}(v,\varphi) + C(\Gamma).$$

Thus by Poincaré's inequality

$$-\int_\Gamma |\theta|^2 \geq -C\left(\int_\Gamma |\nabla_\Gamma\theta_\Gamma|^2 + \mathcal{F}(v,\varphi)\right) - C(\Gamma)$$

and consequently

$$-\int_\Gamma |\nabla_\Gamma \mu|^2 - \int_\Gamma |\nabla_\Gamma \theta|^2 \le -C\left(\int_\Gamma \varepsilon\,|\nabla_\Gamma \varphi_\Gamma|^2 + \frac{1}{\varepsilon}\int_\Gamma W(\varphi) + \frac{\delta}{2}\,|\theta|^2\right) + \tilde{c}. \tag{5.21}$$

Using (5.19) and (5.21), we deduce

$$\frac{d}{dt}\mathcal{F}(v,\varphi) \le C(\delta)\mathcal{F}(v,\varphi)^\beta - C\mathcal{F}(v,\varphi) + \tilde{c}$$

from (5.17). Finally Young's inequality allows us to deduce

$$\frac{d}{dt}\mathcal{F}(v,\varphi) \le C - c\mathcal{F}(v,\varphi),$$

which finishes the proof. □

Corollary 5.9. Assume that Condition 5.6 holds. Then there exists $C > 0$ which depends on the initial data but is independent of t such that for all $t \in [0,\infty)$

$$\mathcal{F}(v(t),\varphi(t)) \le C.$$

Chapter 6

Convergence to the Ohta-Kawasaki equations as $\delta \to 0$

We are now interested in the limit process $\delta \to 0$ for the reduced model in the special case $q(u,v) = c_1 u(1-v) - c_2 v$. If we set $\sigma = \theta_\Gamma - \frac{1}{2}\mu_\Gamma$ and send δ to zero in (2.45)–(2.47) we formally arrive at the limit problem

$$\begin{aligned}
&\partial_t \varphi_\Gamma = \Delta_\Gamma \mu_\Gamma, \\
&\frac{5}{4}\mu_\Gamma = -\varepsilon \Delta_\Gamma \varphi_\Gamma + \frac{1}{\varepsilon} P_\Gamma W'(\varphi_\Gamma) - \frac{1}{2}\sigma, \\
&\Delta_\Gamma \sigma = \frac{c_1 u(t) + c_2}{2} \varphi_\Gamma, \\
&\int_\Gamma \sigma = 0,
\end{aligned}$$

which is a variant of the well-known Ohta-Kawasaki system. The Ohta-Kawasaki equations arise in the modelling of diblock copolymers, see [OK86]. The classical Ohta-Kawasaki system is the H^{-1}–gradient flow of a functional $\mathcal{F}_{OK}$, given by

$$\mathcal{F}_{OK} = \int_\Gamma \frac{\varepsilon}{2} |\nabla_\Gamma \varphi|^2 + \frac{1}{\varepsilon} W(\varphi)\ dx + \left\| \varphi - \fint_\Gamma \varphi \right\|^2_{H^{-1}(\Gamma)}.$$

This functional combines the Ginzburg-Landau energy with the nonlocal contribution in the second summand $\left\|\varphi - \fint_\Gamma \varphi\right\|^2_{H^{-1}(\Gamma)}$. The nonlocal term is motivated by long range interactions between molecules in a system of diblock copolymer molecules. These molecules consist of two different monomers that are chemically bonded to form a linear chain. Usually φ denotes the relative concentration parameter related to these two types of monomers. Even though they are linked together, the two different types of monomers exhibit some repulsion between each other. This does not result in a separation of the diblock copolymer molecule but rather forces the molecules to arrange themselves in configurations that minimize contact between the two different types of monomers without splitting the molecules. These configurations feature different patterns consisting of domains which are rich in one of the two monomers, see e.g. [BF99].

In addition to [OK86], we refer the reader also to [CR03] for a derivation of the Ohta-Kawasaki system and the corresponding energy functional. The resulting Ohta-Kawasaki system is

$$\begin{aligned}
\partial_t \varphi &= \Delta_\Gamma \mu, \\
\mu &= -\varepsilon \Delta_\Gamma \varphi + \frac{1}{\varepsilon} W'(\varphi) - \frac{1}{2}\sigma, \\
\Delta_\Gamma \sigma &= \varphi - \fint \varphi, \\
\fint \sigma &= 0.
\end{aligned}$$

We emphasize that the system we recover in the limit $\delta \searrow 0$ for the reduced model differs slightly from this system and in particular includes the time dependent factor $\frac{c_1 u(t) + c_2}{2}$ in the equation for σ. The function u is given as the solution to the ordinary differential equation

$$\frac{d}{dt}\int_B u(t) = -\int_\gamma q(u,v) = -\frac{c_1}{|B|}\left(\int_B u(t)\right)^2 + \left(c_1 \frac{M - |\Gamma|}{|B|} - c_2\right)\int_B u(t) + c_2 M$$

and due to Remark 2.8, u is bounded for all times if

$$u(0) \in [0, |B|^{-1} M].$$

In particular, $u(t) \to u_\infty$ for $t \to \infty$.

The main result of this section asserts the convergence for $\delta \searrow 0$ rigorously. We remark that the existence of weak solutions to the reduced problem is due to Proposition 4.9.

Proposition 6.1. *Let the exchange term q be given as in* (2.23)*, i.e.*

$$q(u,v) = c_1 u(1-v) - c_2 v.$$

Furthermore, let $\{\delta_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $\lim_{n\to\infty}\delta_n = 0$ and denote by $(u^{\delta_n}, \varphi^{\delta_n}, \mu^{\delta_n}, \theta^{\delta_n}, v^{\delta_n})$ a weak solution to the reduced problem (2.28)–(2.32) *from Proposition 4.9 with $\delta = \delta_n$. We assume that the initial data is independent of δ_n and in addition that the initial data for u belongs to $[0, M|B|^{-1}]$. Then there exists a subsequence (again denoted by $\{\delta_n\}_{n\in\mathbb{N}}$) such that $\{u^{\delta_n}\}_{n\in\mathbb{N}}$ and the mean value free functions $(\varphi_\Gamma^{\delta_n}, \mu_\Gamma^{\delta_n}, \theta_\Gamma^{\delta_n})$ fulfil*

$$\begin{aligned}
u^{\delta_n} &\rightharpoonup u \text{ in } H^1(0,T), \\
\varphi_\Gamma^{\delta_n} &\rightharpoonup \varphi_\Gamma \text{ in } L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma)), \\
\mu_\Gamma^{\delta_n} &\rightharpoonup \mu_\Gamma \text{ in } L^2(0,T;H^1(\Gamma)), \\
\theta_\Gamma^{\delta_n} &\rightharpoonup \theta_\Gamma \text{ in } L^2(0,T;H^1(\Gamma)), \\
\delta_n \partial_t \theta_\Gamma^{\delta_n} &\overset{*}{\rightharpoonup} 0 \text{ in } L^2(0,T;H^{-1}(\Gamma)).
\end{aligned}$$

and such that the limit functions are a weak solution to the modified Ohta-Kawasaki equation, *i.e they fulfil for all $\eta \in L^2(0,T;H^1(\Gamma))$ the equations*

$$
\begin{aligned}
&\frac{d}{dt}\int_B u(t) = -\frac{c_1}{|B|}\left(\int_B u(t)\right)^2 + \left(c_1\frac{M-|\Gamma|}{|B|} - c_2\right)\int_B u(t) + c_2 M \text{ on } (0,T],\\
&\int_0^T\int_\Gamma \partial_t\varphi_\Gamma\eta = -\int_\Gamma \nabla_\Gamma\mu_\Gamma\cdot\nabla_\Gamma\eta,\\
&\frac{5}{4}\int_0^T\int_\Gamma \mu_\Gamma\eta = \int_0^T\int_\Gamma \varepsilon\nabla_\Gamma\varphi_\Gamma\cdot\nabla_\Gamma\eta + \frac{1}{\varepsilon}\int_0^T\int_\Gamma W'(\varphi_\Gamma)\eta - \frac{1}{2}\int_0^T\int_\Gamma \sigma\eta, \text{ and}\\
-&\int_0^T\int_\Gamma \nabla_\Gamma\sigma\cdot\nabla_\Gamma\eta = \int_0^T\int_\Gamma \frac{c_1u(t)+c_2}{2}\varphi_\Gamma\eta,
\end{aligned}
$$

where $\sigma := \theta_\Gamma - \frac{1}{2}\mu_\Gamma$.

The proof relies on the following lemma.

Lemma 6.2. Let (u,φ,v,μ,θ) be a weak solution to the reduced model (2.28)–(2.32). Then the mean value free parts $(u_\Gamma,\varphi_\Gamma,\mu_\Gamma,\theta_\Gamma,v_\Gamma)$ fulfil for all $T<\infty$

$$
\begin{aligned}
\sup_{t\in(0,T)}&\left[\frac{\varepsilon}{2}|\nabla_\Gamma\varphi_\Gamma(t)|^2 + \frac{1}{\varepsilon}W(\varphi_\Gamma(t)) + \frac{\delta}{8}\theta_\Gamma^2(t)\right]\\
&+\|\mu_\Gamma\|^2_{L^2(0,T;H^1(\Gamma))} + \|\theta_\Gamma\|^2_{L^2(0,T;H^1(\Gamma))} \le C(T,\varepsilon,c_2),
\end{aligned}
$$

where $C(T,\varepsilon,c_2)$ depends on the initial data but is independent of δ.

Proof. We multiply equation (2.47) by θ_Γ and integrate over Γ to obtain

$$
\frac{\delta}{8}\frac{d}{dt}\|\theta_\Gamma\|^2_{L^2(\Gamma)} = -\int_\Gamma|\nabla_\Gamma\theta_\Gamma|^2 - \frac{1}{2}\int_\Gamma\partial_t\varphi_\Gamma\theta_\Gamma - \frac{\delta}{4}(c_1u(t)+c_2)\int_\Gamma\theta_\Gamma^2 - \frac{(c_1u(t)+c_2)}{2}\int_\Gamma\varphi_\Gamma\theta_\Gamma \quad (6.1)
$$

Furthermore, multiplying the equation

$$
\mu_\Gamma = -\varepsilon\Delta_\Gamma\varphi + \frac{1}{\varepsilon}P_\Gamma W'(\varphi) - \frac{1}{2}\theta_\Gamma
$$

by $\partial_t\varphi_\Gamma$ and integrating over Γ yields

$$
\frac{1}{2}\int_\Gamma\partial_t\varphi_\Gamma\theta_\Gamma = \int_\Gamma|\nabla_\Gamma\mu_\Gamma|^2 + \frac{d}{dt}\int_\Gamma\left[\frac{\varepsilon}{2}|\nabla_\Gamma\varphi|^2 + \frac{1}{\varepsilon}W(\varphi_\Gamma)\right].
$$

Substituting this into (6.1) implies

$$
\begin{aligned}
\frac{d}{dt}\int_\Gamma\left[\frac{\varepsilon}{2}|\nabla_\Gamma\varphi|^2 + \frac{1}{\varepsilon}W(\varphi_\Gamma) + \frac{\delta}{8}\theta_\Gamma^2\right] &= -\int_\Gamma|\nabla_\Gamma\theta_\Gamma|^2 - \int_\Gamma|\nabla_\Gamma\mu_\Gamma|^2\\
&\quad -\frac{\delta}{4}(c_1u(t)+c_2)\int_\Gamma\theta_\Gamma^2 - \frac{c_1u(t)+c_2}{2}\int_\Gamma\varphi_\Gamma\theta_\Gamma \qquad (6.2)
\end{aligned}
$$

Since $|u(t)|<C$ for all $t\in(0,\infty)$ and some $C>0$ we deduce from Young's inequality for $\beta>0$

$$
\left|\frac{c_1u(t)+c_2}{2}\int_\Gamma\varphi_\Gamma\theta_\Gamma\right| \le C\left(\frac{1}{2\beta}\int_\Gamma\varphi_\Gamma^2 + \frac{\beta}{2}\int_\Gamma\theta_\Gamma^2\right).
$$

Hence Poincaré's inequality implies

$$\left| \frac{c_1 u(t) + c_2}{2} \int_\Gamma \varphi_\Gamma \theta_\Gamma \right| \le C \left(\frac{1}{2\beta} \int_\Gamma \varphi_\Gamma^2 + \frac{\beta}{2} \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 \right).$$

We choose β sufficently small to assure $C(\beta) := 1 - \beta\frac{C}{2} > 0$ Thus (6.2) leads to the inequality

$$\begin{aligned}
&\frac{d}{dt} \int_\Gamma \left[\frac{\varepsilon}{2} |\nabla_\Gamma \varphi|^2 + \frac{1}{\varepsilon} W(\varphi_\Gamma) + \frac{\delta}{8} \theta_\Gamma^2 \right] \\
&\le -C(\beta) \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 - \int_\Gamma |\nabla_\Gamma \mu_\Gamma|^2 - \frac{\delta}{4}(c_1 u(t) + c_2) \int_\Gamma \theta_\Gamma^2 + \frac{C}{\beta} \int_\Gamma \varphi_\Gamma^2 \\
&\le -C(\beta) \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 - \int_\Gamma |\nabla_\Gamma \mu_\Gamma|^2 - \frac{\delta}{4}(c_1 u(t) + c_2) \int_\Gamma \theta_\Gamma^2 + \rho \frac{C}{\beta\varepsilon} \int_\Gamma W(\varphi_\Gamma) + C(\rho, \varepsilon, \beta)
\end{aligned}$$

where we have used Young's inequality with $\rho > 0$ in the second inequality.

By (5.20) we have

$$-C \left(\int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \int_\Gamma |\nabla_\Gamma \mu_\Gamma|^2 \right) \le - \int_\Gamma \left[\frac{\varepsilon}{2} |\nabla_\Gamma \varphi_\Gamma| + \frac{1}{\varepsilon} W(\varphi_\Gamma) \right] + C$$

and for ρ sufficiently small we thus find

$$\begin{aligned}
&\frac{d}{dt} \int_\Gamma \left[\frac{\varepsilon}{2} |\nabla_\Gamma \varphi|^2 + \frac{1}{\varepsilon} W(\varphi_\Gamma) + \frac{\delta}{8} \theta_\Gamma^2 \right] + \frac{C(\beta)}{2} \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \frac{1}{2} \int_\Gamma |\nabla_\Gamma \mu_\Gamma|^2 \\
&\quad \le -C(\beta, \rho, \varepsilon, c_2) \left[\int_\Gamma \frac{\varepsilon}{2} |\nabla_\Gamma \varphi|^2 + \frac{1}{\varepsilon} W(\varphi_\Gamma) + \frac{\delta}{8} \theta_\Gamma^2 \right] + C(\rho, \varepsilon, \beta)
\end{aligned} \tag{6.3}$$

We use the differential form of Gronwall's inequality (see e.g. [Eva10, Appendix B.2(j)]) to deduce

$$\sup_{t \in (0,\infty)} \int_\Gamma \left[\frac{\varepsilon}{2} |\nabla_\Gamma \varphi|^2 + \frac{1}{\varepsilon} W(\varphi_\Gamma) + \frac{\delta}{8} \theta_\Gamma^2 \right] \le C$$

and therefore, after integrating (6.3) in time

$$\begin{aligned}
&\int_\Gamma \left[\frac{\varepsilon}{2} |\nabla_\Gamma \varphi_\Gamma(T)|^2 + \frac{1}{\varepsilon} W(\varphi_\Gamma(T)) + \frac{\delta}{8} \theta_\Gamma^2(T) \right] \\
&\qquad + \frac{C(\beta)}{2} \int_0^T \int_\Gamma |\nabla_\Gamma \theta_\Gamma|^2 + \frac{1}{2} \int_0^T \int_\Gamma |\nabla_\Gamma \mu_\Gamma|^2 \le C(T)
\end{aligned} \tag{6.4}$$

for all $T < \infty$. This proves the assertion of the lemma. □

Based on this uniform estimate we can prove the main result of this section.

Proof of Proposition 6.1. We first observe that the solution u^{δ_n} to equation (2.48) is bounded for all times, see also Remark 2.8(2). Moreover, the equation is independent of δ and the bound is thus also uniform in δ.

By (6.4) we deduce $\delta_n \left\| \theta_\Gamma^{\delta_n} \right\|_{L^2(0,T,L^2(\Gamma))} \le C\delta_n \left\| \nabla_\Gamma \theta_\Gamma^{\delta_n} \right\|_{L^2(0,T,L^2(\Gamma))} \le C\delta_n$, which yields for all $\Psi \in C_c^\infty(\Gamma_T)$

$$\left| \delta_n \int_{\Gamma_T} \theta_\Gamma^{\delta_n} \partial_t \Psi \right| \le \delta_n \left\| \theta_\Gamma^{\delta_n} \right\|_{L^2(0,T,L^2(\Gamma))} \left\| \partial_t \Psi \right\|_{L^2(0,T,L^2(\Gamma))},$$

i.e $\delta_n \partial_t \theta_\Gamma^{\delta_n} \to 0$ in the sense of distributions as $\delta_n \to 0$. At the same time, we can estimate $\|\delta_n \partial_t \theta_\Gamma^{\delta_n}\|_{L^2(0,T;H^{-1}(\Gamma))}$ uniformly in δ_n since by (2.47) for all $\eta \in L^2(0,T;H^1(\Gamma))$ we find

$$\left|\int_0^T \int_\Gamma \delta_n \partial_t \theta_\Gamma^{\delta_n} \eta\right| \le \left|\int_0^T \int_\Gamma \nabla_\Gamma \theta_\Gamma^{\delta_n} \cdot \nabla_\Gamma \eta\right| + \left|\int_0^T \int_\Gamma \frac{1}{2} \nabla_\Gamma \mu_\Gamma^{\delta_n} \cdot \nabla_\Gamma \eta\right| \\ + \left|\int_0^T \int_\Gamma \frac{\delta_n (c_1 u^{\delta_n}(t) + c_2)}{4} \theta_\Gamma^{\delta_n} \eta\right| + \left|\int_0^T \int_\Gamma \frac{c_1 u^{\delta_n}(t) + c_2}{2} \varphi_\Gamma^{\delta_n} \eta\right|,$$

which implies

$$\left|\int_0^T \int_\Gamma \delta_n \partial_t \theta_\Gamma^{\delta_n} \eta\right| \le C \|\eta\|_{L^2(0,T;H^1(\Gamma))}$$

by Lemma 6.2 and the boundedness of $u^{\delta_n}(t)$.

In particular, $\delta_n \partial_t \theta_\Gamma^{\delta_n}$ is bounded in $L^2(0,T;H^{-1}(\Gamma))$. Since $L^2(0,T;H^1(\Gamma))$ is reflexive, its dual space $L^2(0,T;H^1(\Gamma))' = L^2(0,T;H^{-1}(\Gamma))$ is reflexive as well by [Kab11, Theorem 9.12]. Hence there exists a weakly converging subsequence in $L^2(0,T;H^{-1}(\Gamma))$ and some function $\chi \in L^2(0,T;H^{-1}(\Gamma))$ such that $\delta_n \partial_t \theta_\Gamma^{\delta_n} \rightharpoonup \chi$ in $L^2(0,T;H^{-1}(\Gamma))$ as $\delta_n \to 0$. Since χ must coincide with the vanishing distributional limit we deduce $\chi \equiv 0$.

Exploiting equation (2.45), we deduce similarly that $\partial_t \varphi_\Gamma^{\delta_n}$ is bounded uniformly in δ_n in $L^2(0,T;H^{-1}(\Gamma))$. As such, there exists a weakly converging subsequence $\partial_t \varphi_\Gamma^{\delta_n} \rightharpoonup \tilde{\varphi}_\Gamma$.

The bounds in Lemma 6.2 also infer the weak convergence of the mean value free functions $\varphi_\Gamma^{\delta_n}, \theta_\Gamma^{\delta_n}$ and $\mu_\Gamma^{\delta_n}$ in the reflexive space $L^2(0,T;H^1(\Gamma))$. Again, this convergence is meant up to a subsequence.

Calculating the distributional time derivative $\partial_t \varphi_\Gamma^{\delta_n}$ in $D'(0,T;H^1(\Gamma))$ shows $\partial_t \varphi_\Gamma = \tilde{\varphi}_\Gamma$, i.e (after the extraction of a subsequence) we have

$$\varphi_\Gamma^{\delta_n} \rightharpoonup \varphi_\Gamma \text{ in } L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma)).$$

The Aubins-Lions Theorem thus yields $\varphi_\Gamma^{\delta_n} \to \varphi_\Gamma$ in $L^2(0,T;L^2(\Gamma))$ and in particular the pointwise convergence $\varphi_\Gamma^{\delta_n}(p,t) \to \varphi_\Gamma(p,t)$ for almost all $(p,t) \in \Gamma \times (0,T)$. As a direct consequence we deduce $W'(\varphi_\Gamma^{\delta_n}) \to W'(\varphi_\Gamma)$ pointwise almost everywhere on $\Gamma \times (0,T)$. Moreover, the Sobolev embedding theorem assures $\|\varphi_\Gamma^{\delta_n}\|_{L^2(0,T;L^4(\Gamma))} \le \|\varphi_\Gamma^{\delta_n}\|_{L^2(0,T;H^1(\Gamma))} \le C$ uniformly in δ_n. Since $\left|W'(\varphi_\Gamma^{\delta_n})\right| \le C\left(\left|\varphi_\Gamma^{\delta_n}\right|^3 + 1\right)$, we thus find that $W'(\varphi_\Gamma^{\delta_n})$ is bounded uniformly in $L^2(0,T;L^{4/3}(\Gamma))$. Hence there exists a function $\chi \in L^2(0,T;L^{4/3}(\Gamma))$ such that $W'(\varphi_\Gamma^{\delta_n}) \rightharpoonup \chi$ in $L^2(0,T;L^{4/3}(\Gamma))$. Since weak and pointwise limit must coincide, we find

$$W'(\varphi_\Gamma^{\delta_n}) \rightharpoonup W'(\varphi_\Gamma) \text{ in } L^2(0,T;L^{4/3}(\Gamma)).$$

As a result we can pass to the limit in the weak formulations of equations (2.45), (2.46), and (2.47). We obtain for all $\eta \in L^2(0,T;H^1_{(0)}(\Gamma))$

$$\int_0^T \int_\Gamma \partial_t \varphi_\Gamma = -\int_0^T \int_\Gamma \nabla_\Gamma \mu_\Gamma \cdot \nabla_\Gamma \eta$$
$$\int_\Gamma^T \int_\Gamma \left(\mu_\Gamma + \frac{\theta_\Gamma}{2}\right) \eta = \varepsilon \int_0^T \int_\Gamma \nabla_\Gamma \varphi_\Gamma \cdot \nabla_\Gamma \eta + \frac{1}{\varepsilon} \int_0^T \int_\Gamma W'(\varphi_\Gamma) \eta$$
$$-\int_0^T \int_\Gamma \nabla_\Gamma \theta_\Gamma \cdot \nabla_\Gamma \eta + \frac{1}{2} \int_0^T \int_\Gamma \nabla_\Gamma \mu_\Gamma \cdot \nabla_\Gamma \eta = \int_0^T \int_\Gamma \frac{c_1 u(t) + c_2}{2} \varphi_\Gamma \eta.$$

Therefore the limit functions $\varphi_\Gamma, \theta_\Gamma$ and μ_Γ are weak solutions to the problem

$$\begin{aligned}
\partial_t \varphi_\Gamma &= \Delta_\Gamma \mu_\Gamma, \\
\mu_\Gamma + \frac{1}{2}\theta_\Gamma &= -\varepsilon \Delta_\Gamma \varphi_\Gamma + \frac{1}{\varepsilon} P_\Gamma W'(\varphi_\Gamma), \\
\Delta_\Gamma \left(\theta_\Gamma - \frac{1}{2}\mu_\Gamma \right) &= \frac{c_1 u(t) + c_2}{2} \varphi_\Gamma.
\end{aligned}$$

We denote by σ the auxiliary function $\sigma := \theta_\Gamma - \frac{1}{2}\mu_\Gamma$ and find the more familiar formulation

$$\begin{aligned}
&\partial_t \varphi_\Gamma = \Delta_\Gamma \mu_\Gamma, \\
&\frac{5}{4}\mu_\Gamma = -\varepsilon \Delta_\Gamma \varphi_\Gamma + \frac{1}{\varepsilon} P_\Gamma W'(\varphi_\Gamma) - \frac{1}{2}\sigma, \\
&\Delta_\Gamma \sigma = \frac{c_1 u(t) + c_2}{2} \varphi_\Gamma, \\
&\int_\Gamma \sigma = 0
\end{aligned}$$

where again all equations are meant to hold in the weak sense given in Proposition 6.1. □

Chapter **7**

Formal Asymptotics

In this chapter, we start the discussion of the singular limit $\varepsilon \searrow 0$ within the model (2.2) – (2.7). In contrast to the foregoing singular limit $\delta \searrow 0$, the equations do not yield suitable uniform estimates in reflexive spaces, which were necessary to deduce at least some weak compactness as $\delta \searrow 0$.

To overcome theses difficulties, we will focus on techniques from Geometric Measure Theory and derive a convergence result as $\varepsilon \searrow 0$ in the varifold sense. The precise result and its proof will be given in Chapter 8.

As an alternative to the measure theoretic approach, matched asymptotics have been succesfully applied to study singular limits that lead to free boundary problems. While rigorous proofs based on this method are often tedious (see for example [CHL10, ABC94]), suitable assumptions can simplify the necessary steps significantly and allow for the formal derivation of the corresponding limit problem. There is a vast collection of examples throughout the literature in which this method, known as formally matched asymptotics, was successfully applied to formally characterize singular limits. We refer the reader to [Fif88, CF88, AHM08, NMHS99] and [Nay00, KC96] for a general introduction while acknowledging that this is by far not a comprehensive list of references.

Because they yield more information on the limit process, formally matched asymptotics provide a good starting point for the discussion of singular limits. As such a starting point the technique was applied to the model (2.2) – (2.7) by Garcke, Rätz, Röger and the author in [GKRR16, Section 4]. The discussion of the singular limit $\varepsilon \searrow 0$ in Chapter 8 in the measure theoretic setting starts off with a weak formulation of the limit problem, i.e. it builds on some a priori knowledge of the limit problem. Hence we briefly recall the findings from [GKRR16].

The results are only formal since the calculations rely on certain assumptions on the solutions to the diffuse model for $\varepsilon > 0$. Let $(u_\varepsilon, \varphi_\varepsilon, v_\varepsilon, \mu_\varepsilon, \theta_\varepsilon)$ be a solution to (2.2) – (2.7). It is a priori assumed that this solution formally converges to a limit $(u, \varphi, v, \mu, \theta)$ as $\varepsilon \searrow 0$ and that for each $t \in (0, T]$ the zero level set $\{\varphi_\varepsilon(\cdot, t) = 0\}$ converges to a smooth curve $\gamma(t) \subset \Gamma$. The formal asymptotic analysis is now based on the additional assumption that away from the zero level set $\{\varphi_\varepsilon(\cdot, t) = 0\}$, all functions $(u_\varepsilon, \varphi_\varepsilon, v_\varepsilon, \mu_\varepsilon, \theta_\varepsilon)$ admit suitable expansions in ε, i.e.

$$f_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k f_k$$

where $f_\varepsilon = u_\varepsilon, \varphi_\varepsilon, \dots,$ etc. For reason explained later, this expansion is usually referred to as the outer expansion. Note that if f_ε admits such an expansion and $\varepsilon \searrow 0$, the corresponding limit function is then characterized by the coefficient f_0 of order $\mathcal{O}(1)$ in ε.

Information on the coefficients in the outer expansions can be gathered from equations (2.2) – (2.7) by plugging the outer expansions into the equations. Subsequently, we compare the coefficients f_k of the same order in ε.

In particular, collecting all terms of order ε^{-1} in (2.6) implies that

$$W'(\varphi_0) = 0.$$

We thus deduce that away from the curve $\gamma(t)$, the dominant term in the expansion φ_0 only attains the values ± 1.

As a direct consequence, we conclude that in the limit $\varepsilon \searrow 0$ the smooth curve $\gamma(t)$ separates two regions $\Gamma^+(t) := \{\varphi(\cdot,t) = 1\}$ and $\Gamma^-(t) := \{\varphi(\cdot,t) = -1\}$, thus yielding a time dependent partition of Γ. We introduce the notation

$$\mathbf{\Gamma}^{\pm} := \{(x,t) \in \Gamma \times (0,T] : x \in \Gamma^{\pm}(t)\}.$$

As discussed in [GKRR16, Section 4], comparing the coefficients of the same order in ε in the remaining equations leads to the conclusion that away from $\gamma(t)$, i.e. on $\mathbf{\Gamma}^{\pm}$, the limit functions (u,φ,v,μ,θ) must fulfil

$$\begin{aligned}
\varphi &= \pm 1 && \text{on } \mathbf{\Gamma}^{\pm},\\
\partial_t u &= D\Delta u && \text{in } B \times (0,T],\\
-D\nabla u \cdot \nu &= q && \text{on } \Gamma \times (0,T],\\
\Delta_\Gamma \mu &= 0 && \text{on } \mathbf{\Gamma}^{\pm},\\
\partial_t v &= \Delta_\Gamma \theta + q && \text{on } \mathbf{\Gamma}^{\pm},\\
\theta &= \frac{2}{\delta}(2v - 1 \mp 1) && \text{on } \mathbf{\Gamma}^{\pm}.
\end{aligned}$$

So far, we have only discussed the limit process $\varepsilon \searrow 0$ away from the interface $\gamma(t)$. Complementing this discussion with a study of the limit process in a neighbourhood of $\gamma(t)$ eventually yields boundary conditions to the elliptic and parabolic equations on $\mathbf{\Gamma}^{\pm}$ and an evolution equation which governs the evolution of the time-dependent interface $\gamma(t)$.

The convergence $\varphi_\varepsilon \to \pm 1$ heuristically implies that the transition from -1 to $+1$ near $\gamma(t)$ will become steeper as $\varepsilon \searrow 0$, in other words one expects a neighborhood of $\gamma(t)$ where $\nabla_\Gamma \varphi_\varepsilon$ takes large values for small ε. Moreover, one expects the width of this neighborhood to shrink as $\varepsilon \searrow 0$. Usually, this neighborhood is referred to as the transition layer. By introducing new coordinates in a tubular neighborhood of $\gamma(t)$ and rescaling the direction perpendicular to $\gamma(t)$ in ε, the transition layer is diffeomorphic to a fixed domain.

The exact construction of these new coordinates, in particular the treatment of technical difficulties provided by the fact that we are working on a manifold Γ, is given in [GKRR16, Section 4.2]. For the purpose of this brief recapitulation, we restrict ourselves to a simplified view. We choose $s \in [0,L]$ to be the arc length parametrization parameter of $\gamma(t)$ and introduce $z := \frac{d_{\gamma(t)}(x,t)}{\varepsilon}$, where $d_{\gamma(t)}(x,t)$ is the signed distance between $x \in \Gamma$ and $\gamma(t)$ and the spatial scaling ε^{-1} is chosen to account for the rapid transition from -1 to $+1$. It is then possible to find a sufficiently small neighborhood of $\gamma(t)$ which is parametrized by (s,z).

We now assume that on this neighborhood of $\gamma(t)$ all functions have expansions in ε with respect to the new coordinates $(x,t) = \Lambda(s,z,t)$, that is we assume

$$f_\varepsilon(x,t) = F(z,s,t;\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k F_k(z,s,t)$$

where again $f_\varepsilon = \varphi_\varepsilon, v_\varepsilon, \dots$ etc. This expansion is called inner expansion.

Inner and outer expansion are connected through the follwoing matching conditions as $z \to \pm\infty$

$$\begin{aligned} F_0(t,s,\pm\infty) &\sim f_0^\pm(x,t), \\ \partial_z F_0(t,s,\pm\infty) &\sim 0, \\ \partial_z F_1(t,s,\pm\infty) &\sim \nabla_\Gamma f_0^\pm(x,t)\cdot \nu_\gamma, \end{aligned}$$

where $(x,t) = \Lambda(0,s,t)$ and $f_0^\pm(x,t) = \lim_{\delta\to 0} f_0(\exp_x(\pm\delta\nu_\gamma),t)$. We refer the reader to [CF88, GS06] and [Fif88] for a derivation of these matching conditions. After transforming the equations (2.2) – (2.7) to the new coordinates, we plug the inner expansion into the equations and again gather information on the coefficients F_k in the inner expansion by comparing all terms of the same order in ε.

Since any derivative with respect to z in the new coordinates translates to a directional derivative in the direction normal to $\gamma(t)$ and $\partial_t z = -\varepsilon^{-1}\mathcal{V}$, this process leads to equations involving the normal vector ν_γ to $\gamma(t)$ and the normal velocity $\mathcal{V}$ of $\gamma(t)$ given by $\mathcal{V}(x_0,t_0) = \frac{d}{dt}\Big|_{t_0}\gamma_t(s_0)\cdot\nu_\gamma(x_0,t_0)$, see also [DDE05].

This ansatz eventually leads to the following boundary conditions for the above equations on $\mathbf{\Gamma}^\pm$ and evolution equation for the interface $\gamma(t)$

$$\begin{aligned} 2\mu + \theta &= c_0\kappa_g && \text{on } \gamma, \\ [\mu]_-^+ &= 0 && \text{on } \gamma, \\ [\theta]_-^+ &= 0 && \text{on } \gamma, \\ -2\mathcal{V} &= [\nabla_\Gamma \mu]_-^+\cdot\nu_\gamma && \text{on } \gamma, \\ -\mathcal{V} &= [\nabla_\Gamma \theta]_-^+\cdot\nu_\gamma && \text{on } \gamma, \end{aligned}$$

where $[\cdot]_-^+$ is the jump across the interface γ and $\nu_\gamma(x_0,t_0) \in T_{x_0}\Gamma$ denotes the unit normal to $\gamma(t_0)$ in $x_0 \in \gamma(t_0)$, pointing inside $\Gamma^+(t_0)$. The geodesic curvature of $\gamma(t)$ in Γ is denoted by $\kappa_g(\cdot,t)$ and $\mathcal{V}(x_0,t_0)$ denotes the normal velocity of $\gamma(t_0)$ in $x_0 \in \gamma(t_0)$ in direction of $\nu_\gamma(x_0,t_0)$. For the complete calculations, we refer the reader to [GKRR16, Section 4.3].

Chapter 8

The Sharp Interface Limit

The sharp interface model obtained from the formal asymptotic analysis is given by

$$\varphi = \pm 1 \quad \text{on } \mathbf{\Gamma}^{\pm} \times (0,T], \tag{8.1}$$
$$\partial_t u = D\Delta u \quad \text{in } B \times (0,T], \tag{8.2}$$
$$-D\nabla u \cdot \nu = q \quad \text{on } \Gamma \times (0,T], \tag{8.3}$$
$$\Delta_\Gamma \mu = 0 \quad \text{on } \mathbf{\Gamma}^{\pm} \times (0,T], \tag{8.4}$$
$$\partial_t v = \Delta_\Gamma \theta + q \quad \text{on } \mathbf{\Gamma}^{\pm} \times (0,T], \tag{8.5}$$
$$\theta = \frac{2}{\delta}(2v - 1 \mp 1) \text{on } \mathbf{\Gamma}^{\pm} \times (0,T], \tag{8.6}$$
$$2\mu + \theta = c_0 \kappa_g \quad \text{on } \gamma, \tag{8.7}$$
$$[\mu]_-^+ = 0 \quad \text{on } \gamma, \tag{8.8}$$
$$[\theta]_-^+ = 0 \quad \text{on } \gamma, \tag{8.9}$$
$$-2\mathcal{V} = [\nabla_\Gamma \mu]_-^+ \cdot \nu_\gamma \quad \text{on } \gamma, \tag{8.10}$$
$$-\mathcal{V} = [\nabla_\Gamma \theta]_-^+ \cdot \nu_\gamma \quad \text{on } \gamma, \tag{8.11}$$

where $[\cdot]_-^+$ is the jump across the interface γ and $\nu_\gamma(x_0,t_0) \in T_{x_0}\Gamma$ denotes the unit normal to $\gamma(t_0)$ in $x_0 \in \gamma(t_0)$, pointing inside $\Gamma^+(t_0)$. The geodesic curvature of $\gamma(t)$ in Γ is denoted by $\kappa_g(\cdot,t)$ and $\mathcal{V}(x_0,t_0)$ denotes the normal velocity of $\gamma(t_0)$ in $x_0 \in \gamma(t_0)$ in direction of $\nu_\gamma(x_0,t_0)$. For its precise definition, let $\gamma_t : U \to \gamma(t) \subset \Gamma$, $t \in (t_0-\delta, t_0+\delta)$ be a smoothly evolving family of local parameterizations of the curves $\gamma(t)$ by arc length over an open interval $U \subset \mathbb{R}$ and let $\gamma_{t_0}(s_0) = x_0$ for some $s_0 \in U$. Then the normal velocity in (x_0,t_0) is given by

$$\mathcal{V}(x_0,t_0) = \frac{d}{dt}\Big|_{t_0} \gamma_t(s_0) \cdot \nu_\gamma(x_0,t_0),$$

see also [DDE05].

Definition 8.1. Let E be a subset of $\Gamma \times [0,\infty)$ and assume that $\chi_E \in C^0([0,T];L^1(\Gamma)) \cap L^\infty_{w^*}(0,T;BV(\Gamma))$. Consider functions

$$\mu, \theta \in L^2_{loc}([0,T), H^1(\Gamma))$$

and

$$u \in H^1(0,T;H^{-1}(B)) \cap L^2(0,T;H^1(B)), v \in H^1(0,T;H^{-1}(\Gamma)) \cap L^2(0,T;H^1(\Gamma)).$$

Let furthermore V be a Radon measure on $[0,\infty)\times G_1(\Gamma)$ such that V_t is a varifold on Γ for all $t\geq 0$.

We say that the tuple (E,V,u,μ,θ) is a varifold solution to the sharp interface problem (8.1)–(8.11) if for all $T\geq 0$ and for almost every $0\leq\tau\leq t\leq T$ and for all test functions

$$\psi_b\in C_c^\infty([0,T)\times\overline{B}),\psi_s\in C_c^\infty([0,T)\times\Gamma)\text{ and }Y\in C^1(\Gamma,T\Gamma)$$

the following holds:

$$\begin{aligned}
&\int_0^T\int_B u(t,x)\partial_t\psi_b(t,x)\;dx\;dt=\\
&\qquad\int_B u_0(x)\psi_b(0,x)\;dx+\int_0^T\int_B\nabla u(t,x)\cdot\nabla\psi_b(t,x)\;dx\;dt-\int_0^T\int_{\partial B}q\psi_b\;d\mathcal{H}^2\;dt, &&(8.12)\\
&\int_0^T\int_\Gamma -2\chi_{E_t}(p)\partial_t\psi_s(t,p)+\nabla_\Gamma\mu(t,p)\cdot\nabla_\Gamma\psi_s(t,p)\;d\mathcal{H}^2(p)\;dt\\
&\qquad=\int_\Gamma 2\chi_{E_0}(p)\psi_s(0,p)\;d\mathcal{H}^2(p), &&(8.13)\\
&\int_0^T\int_\Gamma -v(t,p)\partial_t\psi_s(t,p)+\nabla_\Gamma\theta(t,p)\cdot\nabla_\Gamma\psi_s(t,p)-\nabla_\Gamma\mu(t,p)\cdot\nabla_\Gamma\psi_s(t,p)\\
&\qquad-q\psi_s(t,p)\;d\mathcal{H}^2(p)\;dt=\int_\Gamma v_0(p)\psi_s(0,p)\;d\mathcal{H}^2(p), &&(8.14)\\
&\qquad\theta=\frac{4}{\delta}\left(2v-2\chi_{E_t}\right), &&(8.15)\\
&\qquad-\langle D\chi_{E_t},(2\mu+\theta)Y\rangle=\langle\delta V_t,Y\rangle, &&(8.16)\\
&\qquad dm_{V_t}(p)\geq 2c_0\left|D\chi_{E_t}\right|(p)\;dp, &&(8.17)\\
&m_{V_t}(\Gamma)+\int_\tau^t\int_\Gamma\left|\nabla_\Gamma\mu(s,p)\right|^2+\left|\nabla_\Gamma\theta(s,p)\right|^2+q\left(\theta(s,p)-u(s,p)\right)\;d\mathcal{H}^2(p)\;ds\leq m_{V_\tau}(\Gamma). &&(8.18)
\end{aligned}$$

Remark 8.2. The concept of a varifold solution given here coincides in the special case that $u=v=0$ with the varifold solutions introduced by Chen in [Che96]. We refer the reader to [Che96, Section 2.4] for a detailed discussion of these solutions and a justification of the definition.

8.1 Main convergence results

Throughout this section, we consider the convergence of weak solutions $(u_\varepsilon,\varphi_\varepsilon,v_\varepsilon,\mu_\varepsilon,\theta_\varepsilon)$ to the diffuse interface problem (2.2)–(2.7) to a weak solution to the sharp interface problem (8.1)–(8.11) as $\varepsilon\searrow 0$. The existence of such solutions $(u_\varepsilon,\varphi_\varepsilon,v_\varepsilon,\mu_\varepsilon,\theta_\varepsilon)$ to the diffuse interface problem is granted by Theorem 4.2 where we proved that solutions belong to the space

$$\mathcal{W}=\mathcal{W}_B\times\mathcal{W}_\Gamma^1\times\mathcal{W}_\Gamma^2\times\mathcal{W}_\Gamma^3\times\mathcal{W}_\Gamma^4$$

where

$$\begin{aligned}
\mathcal{W}_B &:= L^2(0,T;H^1(B)) \cap H^1(0,T;H^{-1}(B)),\\
\mathcal{W}_\Gamma^1 &= L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma)),\\
\mathcal{W}_\Gamma^2 &= L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^{-1}(\Gamma)),\\
\mathcal{W}_\Gamma^3 &= L^2(0,T;H^1(\Gamma)), \text{ and}\\
\mathcal{W}_\Gamma^4 &= L^2(0,T;H^1(\Gamma)).
\end{aligned}$$

Moreover, we suppose that the initial data is well prepared in the following sense.

Condition 8.3 (Well Prepared Initial Data)**.** We assume that there exist constants $C, M, m > 0$ and independent of ε such that the initial data $(u_0^\varepsilon, \varphi_0^\varepsilon, v_0^\varepsilon)$ fulfils

$$\begin{aligned}
\sup_{0<\varepsilon<1}\left[\mathcal{F}(\varphi_0^\varepsilon, v_0^\varepsilon) + \int_B |u_0^\varepsilon|^2 \; dx\right] \le C < \infty,&\\
\int_B u_0^\varepsilon \; dx + \int_\Gamma v_0^\varepsilon \; d\mathcal{H}^2 = M \qquad\qquad & \forall \varepsilon \in (0,1],\\
\frac{1}{|\Gamma|}\int_\Gamma \varphi_0^\varepsilon \; d\mathcal{H}^2 = m \in (-1,1) \;\; & \forall \varepsilon \in (0,1].
\end{aligned}$$

Proposition 8.4. *Assume that q has at most linear growth, i.e. that q fulfils* (2.24)*. Let $T > 0$ and consider initial data that fulfils Condition 8.3 and the corresponding solution $(u_\varepsilon, \varphi_\varepsilon, v_\varepsilon, \mu_\varepsilon, \theta_\varepsilon) \in \mathcal{W}$ to the diffuse interface problem* (2.2)–(2.7)*. Then there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}, \varepsilon_k \to 0$ as $k \to \infty$, such that the following statements are true:*

1. *There exists a set $Q^+ \subset [0,T) \times \Omega$ such that*

   a) $\varphi_{\varepsilon_k}(x,t) \to \varphi(x,t) := \begin{cases} 1 & \text{for } (x,t) \in Q^+ \\ -1 & \text{else} \end{cases}$ *almost everywhere in* $(0,T) \times \Gamma$.

   b) $\varphi_{\varepsilon_k} \to \varphi$ *in* $C^{1/9}\left([0,T]; L^2(\Gamma)\right)$.

   c) $\chi_{Q^+} \in L^\infty_{w^*}(0,T; BV(\Gamma))$.

2. *There exists a function $\mu \in L^2(0,T;H^1(\Gamma))$ such that*
$$\mu_{\varepsilon_k} \rightharpoonup \mu \text{ in } L^2(0,T;H^1(\Gamma)).$$

3. *There exists a function $\theta \in L^2(0,T;H^1(\Gamma))$ such that*
$$\theta_{\varepsilon_k} \rightharpoonup \theta \text{ in } L^2(0,T;H^1(\Gamma)).$$

4. *There exists a function $v \in L^2(0,T;L^2(\Gamma))$ such that*
$$v_{\varepsilon_k} \rightharpoonup v \text{ in } L^2(0,T;L^2(\Gamma)).$$

5. *There exists a function $u \in L^2(0,T;H^1(B)) \cap H^1(0,T;L^2(B))$ such that*
$$u_{\varepsilon_k} \rightharpoonup u \text{ in } L^2(0,T;H^1(B)) \cap H^1(0,T;L^2(B)).$$

Theorem 8.5. *Let $(u, \varphi, v, \mu, \theta)$ be the limit tuple from Proposition 8.4. There exists a Radon measure V on $[0,T] \times G(\Gamma)$ such that the measure $V_t := V(t,\cdot)$ is a varifold for almost all times $t \in [0,T]$. Moreover, the tuple $(\varphi, \mu, v, \theta, u, V)$ is a weak solution to the sharp interface problem* (8.1)–(8.11) *in the varifold sense defined in Definition 8.1.*

8.2 Proof of the main convergence results

8.2.1 Preliminary results

We quickly recall that the exponential map exp_p from differential geometry in a point $p \in \Gamma$ maps the tangent space in p to Γ onto a neighborhood of p. For each $p \in \Gamma$, there is an open neighborhood $W_p \subset T_p\Gamma$ containing zero and an open neighborhood $U \subset \Gamma$ of p such that exp_p restricted to W_p is a diffeomorphism between W_p and U. Since Γ is a compact manifold, there is a real number $r > 0$ such that the ball $B_r(0)$ lies in W_p for all $p \in \Gamma$ and the map exp_p restricted to $B_r(0) \subset \mathbb{R}^2$ is a diffeomorphism onto its image for all $p \in \Gamma$.

Thus the sets $\{exp_p|_{B_{r-\eta}(0)}(B_{r-\eta}(0))\}_{p\in\Gamma}$ form for every $\eta \leq \frac{r}{2}$ a covering of Γ. Since the manifold Γ is compact, there is a finite collection of points $\{p_i\}_{i\in\mathcal{I}}$ such that $\{U_i\}_{i\in\mathcal{I}} := \{exp_{p_i}|_{B_{r-\eta}(0)}(B_{r-\eta}(0))\}_{i\in\mathcal{I}}$ still is a covering of Γ. Together with the maps $\alpha_i : U_i \to B_{r-\eta}(0) \subset \mathbb{R}^n$ defined by $\alpha_i(p) := exp_{p_i}^{-1}(p)$, this covering allows us to define an atlas $\{(U_i, \alpha_i)\}_{i\in\mathcal{I}}$ of Γ.

Observe that for every $i \in \mathcal{I}$ and $x \in \alpha_i(U_i) = B_{r-\eta}(0) \subset \mathbb{R}^n$, the expression $exp_{p_i}(x - \eta y)$ is well defined as long as we assume $y \in B_1(0)$. We will make use of this fact in the following construction of approximating sequences to functions in $L^2(\Gamma)$.

Let ρ be a mollifier satisfying

$$\rho \in C^\infty(\mathbb{R}^2), \qquad 0 \leq \rho \leq 1 \forall x \in \mathbb{R}^2, \qquad \text{supp}(\rho) \subset B_1(0) \text{ and } \int_{\mathbb{R}^2} \rho = 1$$

as usual and introduce the notation $\rho_\eta(x) = \eta^{-2}\rho\left(\frac{x}{\eta}\right)$. Furthermore, let $\{z_i\}_{i\in\mathcal{I}}$ be a partition of unity subordinate to the covering $\{U_i\}_{i\in\mathcal{I}}$ of Γ.

For a function $v \in L^2(\Gamma)$ we can then define the functions $v_i^\eta \in C^\infty(\alpha_i(U_i))$ for every $i \in \mathcal{I}$ and $\eta \leq \frac{r}{2}$ by

$$\begin{aligned} v_i^\eta(x) &:= \left(\rho_\eta * (\alpha_i^{-1,*}(z_i v))\right)(x) \\ &= \int_{B_\eta(0)} \rho_\eta(y)(z_i v)(\alpha_i^{-1}(x-y))\, dy = \int_{B_1(0)} \rho(y)(z_i v)(\alpha_i^{-1}(x-\eta y))\, dy. \end{aligned}$$

We deduce $\operatorname{supp} \alpha^{-1,*}(z_i v) \subset B_{r-\eta}$ from $\operatorname{supp} z_i v \subset U_i$ and since $\operatorname{supp} \rho_\eta \subset B_\eta(0)$ this implies $\operatorname{supp} v_i^\eta \subset B_r(0)$ by the general properties of convolutions. Thus the pullback $\alpha_i^* v_i^\eta$ is well defined for each $i \in \mathcal{I}$ and we can define for $\eta \leq \frac{r}{2}$ the smoothing operator $T_\eta : L^2(\Gamma) \to C^\infty(\Gamma)$ by

$$T_\eta v := \sum_{i\in\mathcal{I}} \alpha_i^* v_i^\eta = \sum_{i\in\mathcal{I}} \alpha_i^* \left(\rho_\eta * (\alpha_i^{-1,*}(z_i v))\right)$$

Lemma 8.6 (Approximation on manifolds)**.** For each $v \in L^2(\Gamma)$, the family $\{v_\eta\}_{0<\eta<r/2}$ defined by $v_\eta := T_\eta v$ is a smooth approximation of v with respect to the L^2-topology, i.e.

$$\|v_\eta - v\|_{L^2(\Gamma)} \to 0 \text{ as } \eta \to 0.$$

Furthermore,

$$\|v_\eta\|_{L^2(\Gamma)} \leq C\, \|v\|_{L^2(\Gamma)} \text{ and} \tag{8.19}$$

$$\|\nabla_\Gamma v_\eta\|_{L^2(\Gamma)} \leq \frac{C}{\eta}\, \|v\|_{L^2(\Gamma)} \tag{8.20}$$

for some constant $C > 0$, independent of η.

Proof. We first prove (8.19). Observe that for all $i \in \mathcal{I}$ we can estimate (see for example [Ste13])

$$\left\| \alpha_i^* v_i^\eta \right\|_{L^2(U_i)} \leq C \left\| v_i^\eta \right\|_{L^2(U_i)}$$

and vice versa

$$\left\| \alpha_i^{-1,*}(z_i v) \right\|_{L^2(\alpha_i(U_i))} \leq C \left\| (z_i v) \right\|_{L^2(U_i)},$$

where the constants C are independent of $i \in \mathcal{I}$ since Γ is compact. Furthermore,

$$\left\| \rho_\eta * (\alpha_i^{-1,*}(z_i v)) \right\|_{L^2(\alpha_1(U_i))} \leq \left\| \rho_\eta \right\|_{L^1(\alpha_1(U_i))} \left\| \alpha_i^{-1,*}(z_i v) \right\|_{L^2(\alpha_1(U_i))} \leq \left\| \alpha_i^{-1,*}(z_i v) \right\|_{L^2(\alpha_1(U_i))}$$

by the usual properties of the convolution. Combining these findings yields (8.19).

The next claim is that $v_\eta \to v$ in $L^2(\Gamma)$. Since $C(\Gamma)$ is a dense subset of $L^2(\Gamma)$, it is sufficient to prove the convergence only for functions $v \in C(\Gamma)$.

One has

$$\sum_{i \in \mathcal{I}} \alpha_i^* \left(\int_{\mathbb{R}^n} \rho_\eta(y) \alpha_i^{-1,*}(z_i) \, dy \right)(p) = 1$$

for all $p \in \Gamma$. Therefore

$$\begin{aligned}
&(v - v_\eta)(p) \\
&\quad = v(p) \sum_{i \in \mathcal{I}} \alpha_i^* \left(\int_{\mathbb{R}^n} \rho_\eta(y) \alpha_i^{-1,*}(z_i) \, dy \right)(p) - \sum_{i \in \mathcal{I}} \alpha_i^* \left(\rho_\eta * \alpha_i^{-1,*}(z_i v) \right)(p) \\
&\quad = \sum_{i \in \mathcal{I}} \left[\int_{\mathbb{R}^2} \rho_\eta(y) \{ (z_i v)(\alpha_i^{-1}(\alpha_i(p))) - (z_i v)(\alpha_i^{-1}(\alpha_i(p) - y)) \} \, dy \right].
\end{aligned}$$

We now split this integral into two parts, namely the integral over a ball of radius ξ around the origin and the integral over all $|y| \geq \xi$. To simplify the expression, we denote the integrand in the last expression by $I_i(y,p)$ and write thereby

$$\begin{aligned}
(v - v_\eta)(p) &= \sum_{i \in \mathcal{I}} \int_{|y| < \xi} I_i(y,p) dy + \sum_{i \in \mathcal{I}} \int_{|y| \geq \xi} I_i(y,p) dy \\
&=: \sum_{i \in \mathcal{I}} A_\eta^i(\xi) + \sum_{i \in \mathcal{I}} B_\eta^i(\xi).
\end{aligned}$$

We now exploit the fact that $\alpha_i^{-1,*}(z_i v) \in C_c(\mathbb{R}^n)$ is uniformly continuous to deduce for every $\varepsilon > 0$ the existence of a radius $\xi_i > 0$ such that

$$\left| A_\eta^i(\xi_i) \right| \leq \sup_{|y| < \xi_i} \left| (z_i v)(\alpha_i^{-1}(\alpha_i(p))) - (z_i v)(\alpha_i^{-1}(\alpha_i(p) - y)) \right| \int_{\mathbb{R}^n} \rho_\eta(y) \, dy \leq \varepsilon.$$

Since the manifold Γ is compact, we can define $\xi_0 > 0$ as the minimum over all ξ_i. By the properties of ρ_η, it is then possible to find $\eta_0 > 0$ such that

$$\left| B_\eta^i(\xi_0) \right| \leq 2 \left\| v \right\|_\infty \int_{|y| \geq \xi_0} \rho_\eta(y) \, dy \leq C\varepsilon$$

for all $\eta < \eta_0$. These two estimates thus imply for every $\varepsilon > 0$ the existence of $\eta_0 > 0$ such that

$$\left\| v - v_\eta \right\|_\infty \leq C\varepsilon \text{ for all } \eta < \eta_0,$$

which is the desired convergence. We now prove the second assertion of the lemma, namely the estimate (8.20). Since

$$\|\nabla_\Gamma v_\eta\|_{L^2(\Gamma)} = \left\|\sum_{i\in\mathcal{I}} \nabla_\Gamma\left(\alpha_i^*\left(\rho_\eta * \alpha_i^{-1,*}(z_i v)\right)\right)\right\|_{L^2(\Gamma)} \tag{8.21}$$

it is sufficient to estimate each summand on the right-hand side in (8.21). We write the gradient on Γ in local coordinates to obtain

$$\left\|\nabla_\Gamma\left(\alpha_i^*\left(\rho_\eta * \alpha_i^{-1,*}(z_i v)\right)\right)\right\|_{L^2(\Gamma)} = \left\|\sum_{k,l}^{2} g^{kl}\frac{\partial}{\partial x_k}\left(\rho_\eta * \alpha_i^{-1,*}(z_i v)\right)\partial_{x_l}\sqrt{g}\right\|_{L^2(\alpha_i(U_i))}$$

and use the fact that all entries in the metric tensor g are bounded, first on each U_i and then by the compactness of Γ on the whole manifold, to see

$$\begin{aligned}
&\left\|\sum_{k,l}^{n} g^{kl}\frac{\partial}{\partial x_k}\left(\rho_\eta * \alpha_i^{-1,*}(z_i v)\right)\partial x_l\sqrt{g}\right\|_{L^2(\alpha_i(U_i))}\\
\leq& C\left\|\sum_{k,l}^{n}\left(\frac{\partial}{\partial_{x_k}}\rho_\eta\right) * \alpha_i^{-1,*}(z_i v)\right\|_{L^2(\alpha_i(U_i))}\\
\leq& \frac{C}{\eta}\left\|\alpha_i^{-1,*}(z_i v)\right\|_{L^2(\alpha_i(U_i))}
\end{aligned}$$

where the last inequality is again due to Young's inequality for convolutions and the chain rule produced the factor $\frac{1}{\eta}$. We use again the estimate

$$\left\|\alpha_i^{-1,*}(z_i v)\right\|_{L^2(\alpha_i(U_i))} \leq C\left\|(z_i v)\right\|_{L^2(U_i)}$$

and deduce inequality (8.20). □

Proposition 8.7. *For every $g \in C^1(\Gamma)$ with $\int_\Gamma g = 0$, there exists a solution $\Psi \in C^2(\Gamma)$ to the problem*

$$\begin{aligned}
\Delta_\Gamma \Psi &= g \qquad \text{on } \Gamma,\\
\int_\Gamma \Psi &= 0
\end{aligned}$$

Furthermore, the estimate

$$\|\Psi\|_{C^2(\Gamma)} \leq C\,\|g\|_{C^1(\Gamma)} \tag{8.22}$$

holds.

Proof. The existence proof relies on the direct method of variational calculus, first finding a weak solution in $H^1(\Gamma) \cap \{\Psi \in L^2(\Gamma) \,|\, \int_\Gamma \Psi\, d\mathcal{H}^2 = 0\}$ by minimizing the functional $I(\Psi) = \int_\Gamma |\nabla_\Gamma \Psi|^2$ before applying regularity theory to deduce that the solution lies in $C^2(\Gamma)$. It can be found in [Aub98]. In order to get the estimate (8.22), one first proves

$$\|\Psi\|_{C^{2,\alpha}(\Gamma)} \leq C\left(\|g\|_{C^{0,\alpha}(\Gamma)} + \|\Psi\|_{C^0(\Gamma)}\right).$$

To this end, consider an atlas $\{(U_i, \alpha_i)\}$ of Γ. Locally on each $\alpha_i(U_i)$, one can then apply the usual Schauder estimates (see [GT01, Chapter 6]). Since the manifold is compact, we can find bounds on the metric the charts such that each estimate carries over to each U_i and therefore to the whole manifold Γ. The following contradiction argument (which also works for the Laplace equation with Neumann boundary values on a domain Ω, see [Nar14] where it is attributed to Lions) then allows us to deduce the estimate (8.22). Assume that (8.22) is wrong. Then for every $k \in \mathbb{N}$, there exist functions $g_k \in C^1(\Gamma)$ with $\int_\Gamma g_k = 0$ such that we have solutions $u_k \in C^{2,\alpha}$ of

$$\begin{aligned} \Delta_\Gamma u_k &= g_k \qquad \text{on } \Gamma, \\ \int_\Gamma u_k &= 0 \end{aligned}$$

satisfying

$$\begin{aligned} &\|u_k\|_{C^{2,\alpha}(\Gamma)} = 1, \text{ and} \\ &\|u_k\|_{C^{2,\alpha}(\Gamma)} > k\left(\|g_k\|_{C^{0,\alpha}(\Gamma)}\right). \end{aligned} \tag{8.23}$$

As a direct consequence, $g_k \to 0$ in $C^{0,\alpha}(\Gamma)$. Since the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $C^{2,\alpha}(\Gamma)$, the embedding properties of Hölder spaces give us a converging subsequence $\{u_k\}_{k\in\mathbb{N}}$ in $C^2(\Gamma)$. If we denote its limit by u_0, this limit has to fulfil

$$\begin{aligned} \Delta_\Gamma u_0 &= 0 \qquad \text{on } \Gamma, \\ \int_\Gamma u_0 &= 0 \end{aligned}$$

and thus $u_0 = 0$. This contradicts (8.23). □

The following lemma will allow us to deduce bounds for $\{\varphi_\varepsilon\}_{\varepsilon>0}$ which are uniform in ε and will play a crucial role in the analysis of the limit process $\varepsilon \searrow 0$.

Lemma 8.8 (Modica-Mortola trick). Let $H : [-1, \infty) \to [0, \infty)$ be defined by

$$H(s) = \int_{-1}^{s} \sqrt{\min\{W(s), 1 + |r|^2\}}\, dr.$$

H is invertible and there are constants $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 |s_1 - s_2|^2 \le |H(s_1) - H(s_2)| \le c_2 |s_1 - s_2| (1 + |s_1| + |s_2|) \tag{8.24}$$

for all $s_1, s_2 \in \mathbb{R}$. Moreover, for any $\varepsilon > 0$ and any solution $(u_\varepsilon, \varphi_\varepsilon, v_\varepsilon, \mu_\varepsilon, \theta_\varepsilon) \in \mathcal{W}$ to (2.2)–(2.7)

$$\sup_{0\le t\le T} \int_\Gamma |\nabla_\Gamma H(\varphi_\varepsilon(x,t))| \le C(T) \tag{8.25}$$

and in particular

$$H(\varphi_\varepsilon) \in L^\infty(0, T; W^{1,1}(\Gamma)).$$

Proof. The existence of constants c_1 and c_2 such that (8.24) holds is a direct consequence from the properties of the double-well potential W. Since the integrand is positive, the function H is strictly monotonically increasing and thus invertible.

We calculate

$$\begin{aligned}\int_\Gamma |\nabla_\Gamma H(\varphi_\varepsilon(x,t))| &= \int_\Gamma \sqrt{W(\varphi_\varepsilon(\cdot,t))}\,|\nabla_\Gamma \varphi_\varepsilon(\cdot,t)| \\ &\le \frac{1}{2}\int_\Gamma \left[\frac{1}{\varepsilon}W(\varphi_\varepsilon(\cdot,t)) + \varepsilon\,|\nabla_\Gamma\varphi_\varepsilon(\cdot,t)|^2\right]\end{aligned}$$

which by the energy estimate in Theorem 4.2 implies (8.25) and $H(\varphi_\varepsilon) \in L^\infty(0,T;W^{1,1}(\Gamma))$. □

8.2.2 Compactness results

Compactness of $\{\varphi_\varepsilon\}$

The main purpose of this section is to prove the convergence of the family $\{\varphi_\varepsilon\}_{\varepsilon\ge 0}$. We expect the limit $\varphi := \lim_{\varepsilon\to 0}\varphi_\varepsilon$ to be a function which takes only the values ±1, although it is not directly clear in which sense this convergence is meant. Nevertheless, it seems to be reasonable to expect the limit φ to be a function of bounded variation. Since this is true for the orginal Cahn-Hilliard equation (see [Che96]), our aim is now to prove this property rigorously for the modified model as well.

Proposition 8.9. *Under the assumptions from Proposition 8.4 there exists a set $Q^+ \subset [0,T)\times\Gamma$ and a subsequence of $\{\varphi_\varepsilon\}_{\varepsilon>0}$ (which we denote by $\{\varphi_{\varepsilon_k}\}$) such that*

1. $\varphi_{\varepsilon_k}(x,t) \xrightarrow{k\to\infty} \varphi(x,t) := \begin{cases} 1 & \textit{for } (x,t)\in Q^+ \\ -1 & \textit{else} \end{cases}$ *almost everywhere in* $(0,T)\times\Gamma$.
2. $\varphi_{\varepsilon_k} \xrightarrow{k\to\infty} \varphi$ *in* $C^{1/9}\left([0,T];L^2(\Gamma)\right)$.
3. $\chi_{Q^+} \in L^\infty_{w^*}(0,T;BV(\Gamma))$.

The proof relies on a suitable compactness argument for the sequence $\{\varphi_\varepsilon\}_{\varepsilon>0}$. So far, the only uniform estimates which allow us to control the sequence $\{\varphi_\varepsilon\}_{\varepsilon>0}$ come from the energy estimate. This estimate however only allows us to deduce

$$\sup_{t\in[0,T]}\left(\varepsilon\,\|\nabla\varphi_\varepsilon(t)\|^2_{L^2(\Omega)}\right) \le C.$$

In the case of the unmodified Cahn-Hiliard equation on a bounded domain in $\mathbb{R}^n$, Chen [Che96] derived estimates which are uniform in ε as a consequence of the Modica-Mortola trick in Lemma 8.8. The following lemma shows the same results for our problem if we modify the proof to account for the fact the we need to work with functions on the manifold Γ.

Lemma 8.10. There exists a positive constant C which is independent of ε such that

$$\sup_{0\le t\le T}\|H(\varphi_\varepsilon(t))\|_{W^{1,1}(\Gamma)} + \|H(\varphi_\varepsilon)\|_{C^{1/8}([0,T);L^1(\Gamma))} + \|\varphi_\varepsilon\|_{C^{1/8}([0,T);L^2(\Gamma))} \le C.$$

Proof. We begin our proof with the estimate for $\|\varphi_\varepsilon\|_{C^{1/8}([0,T);L^2(\Omega))}$. Our aim is to show that

$$\sup_{t,\tau\in[0,T],t\neq\tau} \frac{\left(\int_\Omega |\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau)|^2\ dx\right)^{1/2}}{(t-\tau)^{1/8}} \le C(T). \tag{8.26}$$

Since it is difficult to control this difference between $\varphi_\varepsilon(x,t)$ and $\varphi_\varepsilon(x,\tau)$ directly, we define $\varphi_\varepsilon^\eta := T_\eta\varphi_\varepsilon$ and calculate

$$\begin{aligned}
&\int_\Gamma \left[(\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau))-(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))\right]^2\ dx\\
&=\int_\Gamma (\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau))^2\ dx - 2\int_\Gamma (\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau))(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))\ dx\\
&\quad+\int_\Gamma (\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))^2\ dx
\end{aligned}$$

to obtain

$$\begin{aligned}
&\int_\Gamma (\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))^2\ dx\\
&=\int_\Gamma \left[(\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau))-(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))\right]^2\ dx\\
&\quad+2\int_\Gamma (\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau))(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))\ dx\\
&\quad-\int_\Gamma (\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau))^2\ dx\\
&\le\int_\Gamma \left[(\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon(x,t))+(\varphi_\varepsilon(x,\tau)-\varphi_\varepsilon^\eta(x,\tau))\right]^2\ dx\\
&\qquad+2\int_\Gamma (\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau))(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))\ dx
\end{aligned} \tag{8.27}$$

since $(\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau))^2$ is non-negative. It is therefore sufficient to control the right-hand side above if we want to prove (8.26).

To this end, we first observe that for $0<\alpha<\frac{1}{2}$ and any $t\in(0,T)$

$$\begin{aligned}
&\int_\Gamma\int_\Gamma \frac{|\varphi_\varepsilon(x,t)-\varphi_\varepsilon(y,t)|^2}{d(x,y)^{2+2\alpha}}\ d\mathcal{H}^2(x)\ d\mathcal{H}^2(y)\\
&\le C\int_\Gamma\int_\Gamma \frac{|H(\varphi_\varepsilon(x,t))-H(\varphi_\varepsilon(y,t))|}{d(x,y)^{2+2\alpha}}\ d\mathcal{H}^2(x)\ d\mathcal{H}^2(y).
\end{aligned}$$

by the properties of H discussed in Lemma 8.8. Moreover, for any function f in the Besov space $B^{2\alpha}_{1,1}(\Gamma)$ we have

$$\int_\Gamma\int_\Gamma \frac{|f(x)-f(y)|}{d(x,y)^{2+2\alpha}}\ d\mathcal{H}^2(x)\ d\mathcal{H}^2(y) \le C\|f\|_{B^{2\alpha}_{1,1}(\Gamma)},$$

see Definition 3.13 and Remark 3.14. Since $W^{1,1}(\Gamma)$ embeds in the Besov space $B^{2\alpha}_{1,1}(\Gamma)$ for $0<\alpha<\frac{1}{2}$ and $H(\varphi_\varepsilon(\cdot,t))\in W^{1,1}(\Gamma)$ with $\|H(\varphi_\varepsilon)(\cdot,t)\|_{W^{1,1}(\Gamma)}\le C(T)$ for all $t\in(0,T)$ by

Lemma 8.8, we can choose $f = H(\varphi_\varepsilon(\cdot,t))$ in the inequality above and thus deduce

$$\begin{aligned}&\int_\Gamma\int_\Gamma \frac{|\varphi_\varepsilon(x,t)-\varphi_\varepsilon(y,t)|^2}{d(x,y)^{2+2\alpha}}\ d\mathcal{H}^2(x)\ d\mathcal{H}^2(y)\\ &\le C\,\|H(\varphi_\varepsilon(\cdot,t))\|_{B^{2\alpha}_{1,1}(\Gamma)} \le C\,\|H(\varphi_\varepsilon(\cdot,t))\|_{W^{1,1}(\Gamma)} \le C(T). \end{aligned} \tag{8.28}$$

We refer the reader to Section 3.1 and [Lun09] for more details on Besov spaces while the embedding $W^{1,1}(\Gamma) \hookrightarrow B^{2\alpha}_{1,1}(\Gamma)$ can be found in [Abe12, Corollary 6.14].

Observe that (8.28) also holds for each localization $\alpha_i^{-1,*}(z_i\varphi_\varepsilon)$ of φ_ε where α_i and z_i are defined as in Section 8.2.1. Because of $\int_{\mathbb{R}^2}\rho(y)dy = 1$ and (8.28), we can thus calculate

$$\begin{aligned}&\int_\Gamma |\varphi^\eta_\varepsilon(x,t)-\varphi_\varepsilon(x,t)|^2\ d\mathcal{H}^2(p)\\ &= \int_\Gamma\left|\sum_{i\in\mathcal{I}}\int_{B_1(0)}\rho(y)\left[\alpha_i^{-1,*}(z_i\varphi_\varepsilon)(\alpha_i(p)-\eta y,t)-\alpha_i^{-1,*}(z_i\varphi_\varepsilon)(\alpha_i(p),t)\right]\ dy\right|^2\ d\mathcal{H}^2(p)\\ &\le C\eta^{2+2\alpha}\int_\Gamma\sum_{i\in\mathcal{I}}\int_{B_1(0)}\rho(y)\,|y|^{2+2\alpha}\frac{\left|\alpha_i^{-1,*}(z_i\varphi_\varepsilon)(\alpha_i(p)-\eta y,t)-\alpha_i^{-1,*}(z_i\varphi_\varepsilon)(\alpha_i(p),t)\right|^2}{|\eta y|^{2+2\alpha}}\ dy\ d\mathcal{H}^2(p)\\ &\le C\eta^{2\alpha}\int_\Gamma\sum_{i\in\mathcal{I}}\int_{\mathbb{R}^2}\frac{\left|\alpha_i^{-1,*}(z_i\varphi_\varepsilon)(\alpha_i(p)-\tilde{y},t)-\alpha_i^{-1,*}(z_i\varphi_\varepsilon)(\alpha_i(p),t)\right|^2}{|\tilde{y}|^{2+2\alpha}}\ d\tilde{y}\ d\mathcal{H}^2(p)\\ &\le C(T)\eta^{2\alpha}. \end{aligned} \tag{8.29}$$

by (8.28).

For the next step, we also observe that by (8.20)

$$\|\nabla\varphi^\eta_\varepsilon(\cdot,t)\|_{L^2(\Gamma)} \le C\eta^{-1}\,\|\varphi_\varepsilon(\cdot,t)\|_{L^2(\Gamma)} \le C\eta^{-1}\,\|\varphi_\varepsilon(\cdot,t)\|_{L^4(\Gamma)}$$

and hence

$$\|\nabla\varphi^\eta_\varepsilon(\cdot,t)\|_{L^2(\Gamma)} \le \eta^{-1}C(T) \tag{8.30}$$

by the estimate (2.27).

We did suppose $\varphi_\varepsilon \in \mathcal{W}^1_\Gamma$ which in particular implies $\varphi_\varepsilon \in H^1(0,T;H^{-1}(\Gamma))$. As such the identity

$$\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau) = \int_\tau^t \partial_s\varphi_\varepsilon(x,s)\ ds$$

holds in $H^{-1}(\Gamma)$ for any $0<\tau<t<T$. We use this equation to estimate

$$\begin{aligned}&\left|\int_\Gamma(\varphi^\eta_\varepsilon(x,t)-\varphi^\eta_\varepsilon(x,\tau))\,(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))\ dx\right|\\ &=\left|\langle(\varphi^\eta_\varepsilon(x,t)-\varphi^\eta_\varepsilon(x,\tau)),(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))\rangle_{H^{-1},H^1}\right|\\ &=\left|\int_\tau^t\langle\Delta_\Gamma\mu_\varepsilon(x,s),(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau))\rangle_{H^{-1},H^1}\ ds\right|\\ &=\left|\int_\tau^t\int_\Gamma\nabla\mu_\varepsilon(x,s)\cdot(\nabla\varphi^\eta_\varepsilon(x,t)-\nabla\varphi^\eta_\varepsilon(x,\tau))\ dx\ ds\right|\\ &\le 2\left(\int_\tau^t\|\nabla\mu_\varepsilon\|^2_{L^2(\Gamma)}\right)^{1/2}(t-\tau)^{1/2}\sup_{s\in(0,T)}\|\nabla\varphi^\eta_\varepsilon(\cdot,s)\|_{L^2(\Gamma)} \end{aligned}$$

The energy control in Theorem 4.2 and estimate (8.30) and thus yield

$$\int_\Gamma \left(\varphi_\varepsilon^\eta(x,t)-\varphi_\varepsilon^\eta(x,\tau)\right)\left(\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau)\right)\,dx$$
$$\le C(T)\eta^{-1}(t-\tau)^{1/2} \tag{8.31}$$

Choosing $\alpha=\frac{1}{4}$ and $\eta\le(t-\tau)^{1/2}$, equation (8.27) and the estimates (8.29) and (8.31) yield

$$\begin{aligned}\int_\Gamma |\varphi_\varepsilon(x,t)-\varphi_\varepsilon(x,\tau)|^2\,dx &\le C\left(\eta^{2\alpha}+\eta^{-1}(t-\tau)^{1/2}\right)\\ &\le C(T)(t-\tau)^{1/4}.\end{aligned} \tag{8.32}$$

Consequently, we deduce (8.26).

Since $\sup_{0\le t\le T}\|H(\varphi_\varepsilon(t))\|_{W^{1,1}(\Gamma)}\le C(T)$ follows directly from Lemma 8.8, it only remains to show that $\|H(\varphi_\varepsilon)\|_{C^{1/8}([0,T];L^1(\Gamma))}\le C$. We use (8.24) and estimate (2.27) to deduce

$$\begin{aligned}&\int_\Gamma |H(\varphi_\varepsilon(x,t)-H(\varphi_\varepsilon(x,\tau)|\,dx\\ &\le c_2\|\varphi_\varepsilon(\cdot,t)-\varphi_\varepsilon(\cdot,\tau)\|_{L^2(\Gamma)}\left(|\Gamma|^{1/2}+\|\varphi_\varepsilon(\cdot,t)\|_{L^2(\Gamma)}+\|\varphi_\varepsilon(\cdot,\tau)\|_{L^2(\Gamma)}\right)\\ &\le C\|\varphi_\varepsilon(\cdot,t)-\varphi_\varepsilon(\cdot,\tau)\|_{L^2(\Gamma)}\left(|\Gamma|^{1/2}+\|\varphi_\varepsilon(\cdot,t)\|_{L^4(\Gamma)}+\|\varphi_\varepsilon(\cdot,\tau)\|_{L^4(\Gamma)}\right)\\ &\le C(T)(t-\tau)^{1/8}\end{aligned}$$

from (8.32). □

Proof of Proposition 8.9. According to Lemma 8.10, the function $H(\varphi_\varepsilon(\cdot,t))$ is bounded in $W^{1,1}(\Gamma)$ for every $t\in(0,T)$. Since we study the two-dimensional case, the Rellich-Kondrachov Theorem implies that the embedding $W^{1,1}(\Gamma)\hookrightarrow L^q(\Gamma)$ is compact for every $q\in[1,2)$. Therefore the family $\{H(\varphi_{\varepsilon_k}(x,t))\}_{k\in\mathbb{N}}$ is relatively compact in $L^1(\Gamma)$ for every $t\in(0,T)$. The Arzela-Ascoli theorem (see [Sim87, Lemma 1]) thus implies that the family $\{H(\varphi_{\varepsilon_k}(x,t))\}_{k\in\mathbb{N}}$ is compactly embedded in $C^{1/9}([0,T];L^1(\Gamma))$. Hence there exists a subsequence such that

$$H(\varphi_{\varepsilon_k}(x,t))\to h(x,t)\text{ in }C^{1/9}([0,T];L^1(\Gamma)) \tag{8.33}$$

as $k\to\infty$. We choose another subsequence (named ε_k as well) to conclude that there exists a function $h(x,t)$ such that

$$H(\varphi_{\varepsilon_k}(x,t))\to h(x,t)\text{ almost everywhere in }(0,T)\times\Gamma \tag{8.34}$$

as $k\to\infty$. The relation $h(x,t)=H(\varphi(x,t))$ defines a function $\varphi(x,t)$ since H is strictly monotone and therefore invertible. The first estimate in (8.24) implies immediately that $\varphi_{\varepsilon_k}(x,t)\to\varphi(x,t)$ almost everywhere in $(0,\infty)\times\Gamma$. Since we have $\|\varphi_\varepsilon\|_{C^{1/8}([0,T];L^2(\Gamma))}\le C$ by Lemma 8.10, we can deduce

$$\varphi_{\varepsilon_k}(x,t))\to\varphi(x,t)\text{ in }C^{1/9}([0,T];L^2(\Gamma)$$

in the same way as the convergence of $H(\varphi_{\varepsilon_k}(x,t))$ in (8.33).

We also know that $\int_\Gamma \left((\varphi_{\varepsilon_k})^2 - 1\right)^2 \leq C(T)\varepsilon_k$ since $\frac{1}{\varepsilon_k} W(\varphi_{\varepsilon_k}) \leq C(T)$ by the energy estimate (2.27). That is $(\varphi(x,t))^2 = 1$ almost everywhere and therefore, there exists a set Q^+ such that

$$\varphi = \begin{cases} 1 & \text{for } (x,t) \in Q^+ \\ -1 & \text{everywhere else} \end{cases} \qquad \text{or in other words } \varphi = -1 + 2\chi_{Q^+}.$$

We see directly that as a consequence we have $H(\varphi) = \sqrt{2}\sigma\chi_{Q^+}$ where $\sigma = \int_{-1}^1 \sqrt{\frac{W(s)}{2}}$.

It remains to show that $\chi_{Q^+} \in L^\infty_{w^*}([0,T]; BV(\Gamma))$. The key element will be the lower semicontinuity of the BV norm. First, we observe that we have

$$\int_\Gamma \frac{\varphi_{\varepsilon_k} + 1}{2} = \frac{1}{2}|\Gamma| + \frac{1}{2}\int_\Gamma \varphi_{\varepsilon_k} \leq \frac{1+\varepsilon_k}{2}|\Gamma| + \frac{1}{\varepsilon_k}\int_\Gamma \varphi_{\varepsilon_k}^2.$$

Since $\frac{1}{\varepsilon_k}\int_\Gamma \varphi_{\varepsilon_k}^2$ can be controlled by $\frac{1}{\varepsilon_k}\int_\Gamma W(\varphi_{\varepsilon_k})$, we obtain

$$\int_\Gamma \frac{\varphi_{\varepsilon_k} + 1}{2} \leq C(T) \text{ for all } \varepsilon_k < 1$$

from (2.27). The lower semicontinuity of the BV norm allows us then to deduce

$$|\chi_{Q^+}| = \int_\Gamma \frac{\varphi + 1}{2} \leq \int_\Gamma \frac{\varphi_{\varepsilon_k} + 1}{2} \leq C(T)$$

for every t. Furthermore, we have $|DH(\varphi_{\varepsilon_k}(\cdot,t))|(\Gamma) \leq C(T)$ for every t by (8.25). Hence we obtain

$$|D\chi_{Q^+}|(\Gamma) = \frac{1}{\sqrt{2}\sigma}|DH(\varphi(\cdot,t))|(\Gamma) \leq \frac{1}{\sqrt{2}\sigma}C(T)$$

by the lower semicontinuity of the BV norm. □

Weak compactness of $\{\mu_\varepsilon\}$

Lemma 8.11. There exist constants $C > 0$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ and all $t \in [0,T)$ the estimate

$$\|\mu_\varepsilon(t)\|_{H^1(\Gamma)} \leq C\left(\int_\Gamma \frac{\varepsilon}{2}|\nabla_\Gamma\varphi_\varepsilon(t)|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon(t))\, d\mathcal{H}^2 + \|\nabla_\Gamma\mu_\varepsilon(\cdot,t)\|_{L^2(\Gamma)} + \|\theta_\varepsilon(\cdot,t)\|_{L^2(\Gamma)}\right) \tag{8.35}$$

holds.

Remark 8.12. The proof follows the proof of the weak compactness result for the chemical potential proved by Chen in [Che96] for the Cahn-Hilliard equation in a bounded smooth domain in $\mathbb{R}^n$. As the reader will see, the proof of Lemma 8.11 differs from his proof mainly in the technical difficulties coming from the manifold setting we are working in. Most of these difficulties were already addressed in Section 8.2.1 above.

Proof. By Poincaré's inequality and the triangle inequality,

$$\begin{aligned}\|\mu_\varepsilon(t)\|_{L^2(\Gamma)} &\leq \|\mu_\varepsilon(t) - \overline{\mu}_\varepsilon(t)\|_{L^2(\Gamma)} + |\overline{\mu}_\varepsilon| \\ &\leq C\|\nabla_\Gamma\mu_\varepsilon(t)\|^2_{L^2(\Gamma)} + \left|\overline{(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon)(t)}\right| + C\|\theta_\varepsilon(t)\|^2_{L^2(\Gamma)}\end{aligned}$$

where $\overline{f}(t)$ denotes the mean value of the function f over Γ. For the sake of convenience we simplify our notation: We define $\omega_\varepsilon(t) := \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)(t)$ and write in most cases ω_ε instead of $\omega_\varepsilon(t)$. Hence it is sufficient to control the mean value of ω_ε if we want to prove (8.35). Let Y be any tangential C^1 vector field on Γ and test equation (2.5) by $Y \cdot \nabla_\Gamma \varphi_\varepsilon$. The resulting equation reads

$$\begin{aligned}\int_\Gamma Y \cdot \nabla_\Gamma \varphi_\varepsilon \omega_\varepsilon \, d\mathcal{H}^2 &= \int_\Gamma Y \cdot \nabla_\Gamma \varphi_\varepsilon \left(-\varepsilon \Delta_\Gamma \varphi + \varepsilon^{-1} W'(\varphi_\varepsilon)\right) \, d\mathcal{H}^2 \\ &= -\int_\Gamma \nabla_\Gamma Y : \left(\left(\frac{\varepsilon \left|\nabla \varphi_\varepsilon(x,t)\right|^2}{2} + \frac{W(\varphi_\varepsilon(x,t))}{\varepsilon}\right)\mathrm{Id} \right.\\ &\quad \left. - \varepsilon \nabla \varphi_\varepsilon(x,t) \otimes \nabla \varphi_\varepsilon(x,t)\right) \, d\mathcal{H}^2. \end{aligned} \tag{8.36}$$

An integration by parts for the left hand side yields

$$\begin{aligned}\int_\Gamma Y \cdot \nabla_\Gamma \varphi_\varepsilon \omega_\varepsilon \, d\mathcal{H}^2 = &-\int_\Gamma Y \cdot \nabla_\Gamma \omega_\varepsilon \varphi_\varepsilon \, d\mathcal{H}^2 - \int_\Gamma \left(\omega_\varepsilon - \overline{\omega_\varepsilon}\right) \varphi_\varepsilon \operatorname{div}_\Gamma Y \, d\mathcal{H}^2 \\ &- \overline{\omega}_\varepsilon \int_\Gamma \varphi_\varepsilon \operatorname{div}_\Gamma Y \, d\mathcal{H}^2.\end{aligned}$$

In turn, the mean value of ω_ε can be expressed as

$$\begin{aligned}\overline{\omega}_\varepsilon = &\frac{1}{\int_\Gamma \varphi_\varepsilon \operatorname{div}_\Gamma Y \, d\mathcal{H}^2} \left\{ \int_\Gamma \nabla_\Gamma Y : \left(\left(\frac{\varepsilon \left|\nabla \varphi_\varepsilon(x,t)\right|^2}{2} + \frac{W(\varphi_\varepsilon(x,t))}{\varepsilon}\right)\mathrm{Id}\right.\right. \\ &\left.\left. - \varepsilon \nabla \varphi_\varepsilon(x,t) \otimes \nabla \varphi_\varepsilon(x,t)\right) - \varphi_\varepsilon Y \cdot \nabla_\Gamma \omega_\varepsilon - \varphi_\varepsilon \operatorname{div}_\Gamma Y \left(\omega_\varepsilon - \overline{\omega}_\varepsilon\right) \, d\mathcal{H}^2 \right\}.\end{aligned} \tag{8.37}$$

As in [Che96], the proof now relies on a clever choice of the vector field Y. To this end, let $\{\varphi_{\eta,\varepsilon}\}_{\eta>0} \subset C^\infty(\Gamma)$ be the family of functions given by $\varphi_{\eta,\varepsilon} = T_\eta \varphi_\varepsilon$. In particular, $\varphi_{\eta,\varepsilon}$ approximates φ_ε and the estimates from Lemma 8.6 are fulfiled. We then define Ψ to be the solution to

$$\begin{aligned}\Delta_\Gamma \Psi &= \varphi_{\eta,\varepsilon} - \overline{\varphi}_{\eta,\varepsilon} \qquad \text{on } \Gamma, \\ \int_\Gamma \Psi &= 0.\end{aligned}$$

By Proposition 8.7 this solution exists and the estimate from the proposition yields

$$\left\|\Psi\right\|_{C^2(\Gamma)} \leq C \left\|\varphi_{\eta,\varepsilon}\right\|_{C^1(\Gamma)}. \tag{8.38}$$

Since

$$K := \sup_{p \in \Gamma} \left|T_\eta(1)(p)\right| < \infty$$

we can estimate

$$\begin{aligned}
\|\varphi_{\eta,\varepsilon}\|_{C(\Gamma)} &\le K + \sup_{p\in\Gamma} |\varphi_{\eta,\varepsilon}(p) - (T_\eta 1)(p)| \\
&\le K + \sup_{p\in\Gamma}\left[\sum_{i\in\mathcal{I}} \int_{B_\eta(0)} \rho_\eta(y)\alpha_i^{-1,*}\,|(z_i(|\varphi_\varepsilon|-1))(\alpha_i(p)-y)|\ dy\right] \\
&\le K + \sup_{p\in\Gamma}\left[\sum_{i\in\mathcal{I}} \left(\int_{B_\eta(0)} \rho_\eta^2(y)\right)^{1/2}\left(\int_{B_\eta(0)} (z_i(|\varphi_\varepsilon|-1))^2(\alpha_i^{-1}(\alpha_i(p)-y))\ dy\right)^{1/2}\right] \\
&\le K + \frac{C(T)}{\eta^{n/2}}\,\||\varphi_\varepsilon|-1\|_{L^2(\Gamma)} \le K + C(T)\varepsilon^{1/2}\eta^{-n/2}.
\end{aligned}$$

A similarly tedious calculation (see also the proof of Lemma 8.6) shows then

$$\|\varphi_{\eta,\varepsilon}\|_{C^1(\Gamma)} \le C\eta^{-1}(1 + C(T)\varepsilon^{1/2}\eta^{n/2}).$$

Thus (8.38) implies

$$\|\Psi\|_{C^2(\Gamma)} \le C\,\|\varphi_{\eta,\varepsilon}\|_{C^1(\Gamma)} \le C\eta^{-1}(1 + C\varepsilon^{1/2}\eta^{n/2}).$$

Returning to (8.37), we now choose Y to be given as $Y = \nabla_\Gamma\Psi$. Therefore the numerator on the right hand side in (8.37) can be estimated by

$$\begin{aligned}
&\left|\int_\Gamma \nabla_\Gamma Y : \left(\left(\frac{\varepsilon\,|\nabla_\Gamma\varphi_\varepsilon(x,t)|^2}{2} + \frac{W(\varphi_\varepsilon(x,t))}{\varepsilon}\right)\mathrm{Id}\right.\right. \\
&\qquad \left.\left. - \varepsilon\nabla_\Gamma\varphi_\varepsilon(x,t)\otimes\nabla_\Gamma\varphi_\varepsilon(x,t)\right) - \varphi_\varepsilon\nabla_\Gamma\Psi\cdot\nabla_\Gamma\omega_\varepsilon - \varphi_\varepsilon\Delta_\Gamma\Psi\,(\omega_\varepsilon - \overline{\omega}_\varepsilon)\ d\mathcal{H}^2\right| \\
&\qquad \le C\,\|\Psi\|_{C^2(\Gamma)}\left[\mathcal{F}(\varphi_\varepsilon, v_\varepsilon) + \|\varphi_\varepsilon\|_{L^2(\Gamma)}\,\|\nabla_\Gamma\omega_\varepsilon\|_{L^2(\Gamma)} + \|\varphi_\varepsilon\|_{L^2(\Gamma)}\,\|\omega_\varepsilon - \overline{\omega_\varepsilon}\|_{L^2(\Gamma)}\right] \\
&\qquad \le C(T)\eta^{-1}\left(1 + \varepsilon^{1/2}\eta^{n/2}\right)\left(\mathcal{F}(\varphi_\varepsilon, v_\varepsilon) + \|\nabla_\Gamma\omega_\varepsilon\|_{L^2(\Gamma)}\right).
\end{aligned}$$

It remains thus to show a lower bound for the denominator in (8.37). To this end, we directly calculate

$$\begin{aligned}
\int_\Gamma \Delta_\Gamma\Psi\varphi_\varepsilon\ d\mathcal{H}^2 &= \int_\Gamma (\varphi_{\eta,\varepsilon} - \overline{\varphi_{\eta,\varepsilon}})\,\varphi_\varepsilon\ d\mathcal{H}^2 \\
&= \int_\Gamma (\varphi_{\eta,\varepsilon} - \varphi_\varepsilon)\,\varphi_\varepsilon\ d\mathcal{H}^2 + \int_\Gamma \left(\varphi_\varepsilon^2 - 1\right)\ d\mathcal{H}^2 + |\Gamma|\left(1 - \overline{\varphi}_\varepsilon^2\right) \\
&\quad + |\Gamma|\,\overline{\varphi}_\varepsilon\left(\overline{\varphi}_\varepsilon - \overline{\varphi}_{\eta,\varepsilon}\right).
\end{aligned}$$

Next we observe that there exists a $C \in \mathbb{R}$ such that for all $s \in \mathbb{R}$ the double-well potential W fulfils $s^2 \le C\,(W(s) + 1)$. Hence the energy estimate implies $\|\varphi_\varepsilon\|_{L^2(\Gamma)} \le C + \varepsilon C(T)$ as well as $\int_\Gamma |\varphi_\varepsilon^2 - 1| \le C\int_\Gamma W(\varphi_\varepsilon) \le \varepsilon C(T)$. Given the fact that $\overline{\varphi}_\varepsilon = m_0 \in (-1,1)$ for all $t \in (0,T)$ we fix some $\varepsilon_1 > 0$ and derive the lower bound

$$\begin{aligned}
\left|\int_\Gamma \Delta_\Gamma\Psi\varphi_\varepsilon\ d\mathcal{H}^2\right| &= \left|\int_\Gamma (\varphi_{\eta,\varepsilon} - \overline{\varphi_{\eta,\varepsilon}})\,\varphi_\varepsilon\ d\mathcal{H}^2\right| \\
&\ge |\Gamma|\,(1 - m_0^2) - C(T,\varepsilon_1)\,\|\varphi_{\eta,\varepsilon} - \varphi_\varepsilon\|_{L^2(\Gamma)} - \int_\Gamma |\varphi_\varepsilon^2 - 1| - |\Gamma|\,m_0\,|\overline{\varphi}_\varepsilon - \overline{\varphi}_{\eta,\varepsilon}| \\
&\ge |\Gamma|\,(1 - m_0^2) - C(T,\varepsilon_1)\,\|\varphi_{\eta,\varepsilon} - \varphi_\varepsilon\|_{L^2(\Gamma)} - \varepsilon C(T) - |\Gamma|\,m_0\,|\overline{\varphi}_\varepsilon - \overline{\varphi}_{\eta,\varepsilon}|
\end{aligned}$$

for all $0 < \varepsilon \le \varepsilon_1$. By Proposition 8.9 we know $\varphi_{\varepsilon_k} \to \varphi$ in $C^{1/9}([0,T]; L^2(\Gamma))$ and by Lemma 8.6 the operators $T_\eta : L^2(\Gamma) \to L^2(\Gamma)$ are linear and uniformly bounded in η. As a consequence

$$\begin{aligned} \|\varphi_{\eta,\varepsilon} - \varphi_\varepsilon\|_{L^2(\Gamma)} &\le \|\varphi_{\eta,\varepsilon} - \varphi_\eta\|_{L^2(\Gamma)} + \|\varphi_\eta - \varphi\|_{L^2(\Gamma)} + \|\varphi - \varphi_\varepsilon\|_{L^2(\Gamma)} \\ &\le C\,\|\varphi_\varepsilon - \varphi\|_{L^2(\Gamma)} + \|\varphi_\eta - \varphi\|_{L^2(\Gamma)} + \|\varphi - \varphi_\varepsilon\|_{L^2(\Gamma)} \end{aligned}$$

uniformly in $t \in [0,T]$. This implies that we can thus find a (possibly small) $\eta > 0$ and $0 < \varepsilon_0 \le \varepsilon_1$ such that

$$\int_\Gamma \Delta_\Gamma \Psi \varphi_\varepsilon \; d\mathcal{H}^2 \ge \frac{1}{2} |\Gamma| \,(1 - m_0^2)$$

for all $0 < \varepsilon < \varepsilon_0$. Combining the upper bound on the numerator and the lower bound on the denominator in (8.37), we find

$$|\overline{\omega_\varepsilon}| \le \frac{2C\eta^{-1}\left(1 + \varepsilon^{1/2}\eta^{n/2}\right)\left(\mathcal{F}(\varphi_\varepsilon, v_\varepsilon) + \|\nabla_\Gamma \omega_\varepsilon\|_{L^2(\Gamma)}\right)}{|\Gamma|\,(1 - m_0^2)},$$

which proves the lemma. □

8.2.3 Proof of the upper bound for the discrepancy measure

The aim of this section is to prove the following upper bound for the positive part of the discrepancy measure $\xi^\varepsilon(\varphi_\varepsilon)$, which is defined as

$$\xi^\varepsilon(\varphi_\varepsilon) := \left(\frac{\varepsilon}{2} |\nabla_\Gamma \varphi_\varepsilon|^2 - \frac{1}{\varepsilon} W(\varphi_\varepsilon)\right)$$

Proposition 8.13. *There exists a positive constant $\eta_0 \in (0,1]$ and continuous, non-increasing and positive functions M_1 and M_2 defined on $(0,\eta_0]$ such that for every $\eta \in (0,\eta_0]$, every $\varepsilon \in \left(0, \frac{1}{M_1(\eta)}\right]$ and every tuple $(\varphi_\varepsilon, v_\varepsilon, u_\varepsilon, \mu_\varepsilon, \theta_\varepsilon) \in \mathcal{W}$ which solves (2.2)–(2.7) the estimate*

$$\begin{aligned} &\int_\Gamma (\xi^\varepsilon(\varphi_\varepsilon))_+ \; d\mathcal{H}^2 \\ &\quad \le \eta \int_\Gamma \frac{\varepsilon}{2} |\nabla \varphi|^2 + \varepsilon^{-1} W(\varphi) \; d\mathcal{H}^2 + \varepsilon M_2(\eta) \int_\Gamma \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 \; d\mathcal{H}^2 \end{aligned} \tag{8.39}$$

holds.

This lemma was proved by Chen for the Cahn-Hilliard equation in [Che96]. We will adapt his proof, modifying it where it is necessary. It consists of several steps, which correspond to the lemmas below. The basic idea is to study the localized equations in each chart (and thus studying equations in the Euclidean space, as Chen did) and to prove Proposition 8.13 by a blow up argument. The core of the proof lies in the following Lemma 8.14 and its application in the blow-up argument used later in the proof of Proposition 8.13.

Lemma 8.14 ([Che96, Lemma 4.1])**.** Assume that $\Phi \in W^{1,2}_{loc}(\mathbb{R}^n)$ satisfies the equation

$$\Delta \Phi = W'(\Phi) \text{ in } \mathbb{R}^n.$$

Then $\Phi \in C^3(\mathbb{R}^n)$, $-1 \le \Phi \le 1$ in $\mathbb{R}^n$, and

$$\frac{1}{2} |\nabla \Phi(x)|^2 \le W(\Phi(x)) \;\; \forall x \in \mathbb{R}^n.$$

Proof. This is exactly Lemma 4.1 in [Che96]. □

Remark 8.15 ([Che96, Lemma 4.2])**.** For the sake of completeness, let us remark that the assertion of Lemma 8.14 remains true if $\Phi \in W^{1,2}_{loc}(\mathbb{R}^{n-1} \times [0,\infty))$ satisfies the equation in the halfspace, i.e. if

$$\begin{aligned} \Delta\Phi &= W'(\Phi) && \text{in } \mathbb{R}^{n-1} \times [0,\infty) \\ \frac{\partial}{\partial n}\Phi &= 0 && \text{on } \mathbb{R}^{n-1} \times \{0\}. \end{aligned}$$

The proof can be found in [Che96]. This variant of Lemma 8.14 allows for a discussion of the Cahn-Hilliard equation on bounded domains in $\mathbb{R}^n$. Since we work on a compact manifold without boundary, we only need to use Lemma 8.14 here.

The outline of the proof of Proposition 8.13 now is as follows: In Lemma 8.16 we prove an estimate which allows us to control $\mathcal{F}(v,\varphi)$ away from the interface between the regions $\{\varphi = 1\}$ and $\{\varphi = -1\}$. We will then introduce rescaled coordinates on the manifold Γ and prove Lemma 8.17 which gives a localized version of estimate (8.39) in these coordinates under the assumption that μ_ε and θ_ε are sufficiently small. The proof of this lemma will be based on Lemma 8.14. Finally, it will be possible to combine the local results in Lemma 8.16 and Lemma 8.17 to derive estimate (8.39) on the entire manifold Γ.

Hence we start with an estimate on $\mathcal{F}(v,\varphi)$ in the regions away from the interface.

Lemma 8.16. There exist positive constants $C_0 > 0$ and $\eta_0 > 0$ such that for every $\eta \in (0,\eta_0]$, every $0 < \varepsilon \le 1$ and for every tupel $(\varphi_\varepsilon, v_\varepsilon, u_\varepsilon, \mu_\varepsilon, \theta_\varepsilon) \in \mathcal{W}$ which solves problem (2.2)–(2.7) the estimate

$$\begin{aligned} &\int_{\{p\in\Gamma \mid |\varphi_\varepsilon|\ge 1-\eta\}} \frac{\varepsilon}{2}|\nabla\varphi_\varepsilon(p)|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon(p)) + \frac{1}{\varepsilon}\left(W'(\varphi_\varepsilon(p))\right)^2 \, d\mathcal{H}^2(p) \\ \le & C_0\eta \int_{\{p\in\Gamma \mid |\varphi_\varepsilon|\le 1-\eta\}} \varepsilon\,|\nabla_\Gamma\varphi_\varepsilon(p)|^2 \, d\mathcal{H}^2(p) + C_0\varepsilon \int_\Gamma \left(\mu_\varepsilon(p) + \frac{1}{2}\theta_\varepsilon(p)\right)^2 \, d\mathcal{H}^2(p) \end{aligned}$$

holds.

Proof. Given the form of W, there exists a constant $c_0 > 0$ such that $W''(r) \ge c_0 |r|^2$ for all $|r| \ge 1 - c_0$. For any $\eta \in (0, c_0/2)$ we now define a function g by

$$g(s) := \begin{cases} W'(s) & \text{for } |s| \ge 1-\eta \\ 0 & \text{for } |s| \le 1-c_0 \end{cases}$$

and affine linear in between. Testing (2.5) with $g(\varphi_\varepsilon)$ thus yields

$$\int_\Gamma \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right) g(\varphi_\varepsilon)\, d\mathcal{H}^2 = \int_\Gamma \left(\varepsilon g'(\varphi_\varepsilon)\,|\nabla_\Gamma\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}W'(\varphi_\varepsilon)g(\varphi_\varepsilon)\right) d\mathcal{H}^2.$$

Since by the definition of g we know

$$\begin{aligned} \left|\int_\Gamma \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right) g(\varphi_\varepsilon)\, d\mathcal{H}^2\right| &\le \int_\Gamma \frac{\varepsilon}{2}\left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 + \frac{1}{2\varepsilon} g(\varphi_\varepsilon)^2 \, d\mathcal{H}^2 \\ &\le \int_\Gamma \frac{\varepsilon}{2}\left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 + \frac{1}{2\varepsilon} g(\varphi_\varepsilon)W'(\varphi_\varepsilon)\, d\mathcal{H}^2, \end{aligned}$$

we thus deduce

$$
\begin{aligned}
&\int_{\Gamma\cap\{|\varphi_\varepsilon|\geq 1-\eta\}} \left(\varepsilon W''(\varphi_\varepsilon)\left|\nabla_\Gamma\varphi_\varepsilon\right|^2 + \frac{1}{\varepsilon}(W'(\varphi_\varepsilon))^2\right)\, d\mathcal{H}^2 \\
=&\int_{\Gamma\cap\{|\varphi_\varepsilon|\geq 1-\eta\}} \left(\varepsilon g'(\varphi_\varepsilon)\left|\nabla_\Gamma\varphi_\varepsilon\right|^2 + \frac{1}{\varepsilon}W'(\varphi_\varepsilon)g(\varphi_\varepsilon)\right)\, d\mathcal{H}^2 \\
\leq&\int_\Gamma \left(\frac{\varepsilon}{2}\left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 + \frac{1}{2\varepsilon}W'(\varphi_\varepsilon)g(\varphi_\varepsilon)\right)\, d\mathcal{H}^2 \\
&-\int_{\Gamma\cap\{|\varphi_\varepsilon|<1-\eta\}} \left(\varepsilon g'(\varphi_\varepsilon)\left|\nabla_\Gamma\varphi_\varepsilon\right|^2 + \frac{1}{\varepsilon}W'(\varphi_\varepsilon)g(\varphi_\varepsilon)\right)\, d\mathcal{H}^2 \\
=&\int_\Gamma \frac{\varepsilon}{2}\left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2\, d\mathcal{H}^2 - \int_{\Gamma\cap\{|\varphi_\varepsilon|<1-\eta\}} \left(\varepsilon g'(\varphi_\varepsilon)\left|\nabla_\Gamma\varphi_\varepsilon\right|^2\right)\, d\mathcal{H}^2 \\
&-\int_{\Gamma\cap\{|\varphi_\varepsilon|<1-\eta\}} \frac{1}{2\varepsilon}W'(\varphi_\varepsilon)g(\varphi_\varepsilon)\, d\mathcal{H}^2 + \int_{\Gamma\cap\{|\varphi_\varepsilon|\geq 1-\eta\}} \frac{1}{2\varepsilon}W'(\varphi_\varepsilon)g(\varphi_\varepsilon)\, d\mathcal{H}^2.
\end{aligned}
$$

We absorb the last term on the right-hand side which results in

$$
\begin{aligned}
&\int_{\Gamma\cap\{|\varphi_\varepsilon|\geq 1-\eta\}} \left(\varepsilon W''(\varphi_\varepsilon)\left|\nabla_\Gamma\varphi_\varepsilon\right|^2 + \frac{1}{2\varepsilon}(W'(\varphi_\varepsilon))^2\right)\, d\mathcal{H}^2 \\
\leq&\int_\Gamma \frac{\varepsilon}{2}\left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2\, d\mathcal{H}^2 - \int_{\Gamma\cap\{|\varphi_\varepsilon|<1-\eta\}} \left(\varepsilon g'(\varphi_\varepsilon)\left|\nabla_\Gamma\varphi_\varepsilon\right|^2\right)\, d\mathcal{H}^2 \\
&-\int_{\Gamma\cap\{|\varphi_\varepsilon|<1-\eta\}} \frac{1}{2\varepsilon}W'(\varphi_\varepsilon)g(\varphi_\varepsilon)\, d\mathcal{H}^2.
\end{aligned}
$$

Moreover, $W'g \geq 0$ as $g(s) = W'(s)$ for $|s| \geq 1-\eta$ or $W'(s)g(s) = 0$ for $|s| \leq 1-c_0$ and $W'(s)$ and $g(s)$ have the same sign everywhere else. As such, we can neglect the last term above, and obtain

$$
\begin{aligned}
&\int_{\Gamma\cap\{|\varphi_\varepsilon|\geq 1-\eta\}} \left(\varepsilon W''(\varphi_\varepsilon)\left|\nabla_\Gamma\varphi_\varepsilon\right|^2 + \frac{1}{2\varepsilon}(W'(\varphi_\varepsilon))^2\right)\, d\mathcal{H}^2 \\
\leq&\frac{\varepsilon}{2}\int_\Gamma \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2\, d\mathcal{H}^2 - \int_{\Gamma\cap\{|\varphi_\varepsilon|\leq 1-\eta\}} \varepsilon g'(\varphi_\varepsilon)\left|\nabla_\Gamma\varphi_\varepsilon\right|^2\, d\mathcal{H}^2.
\end{aligned}
\tag{8.40}
$$

Observe that

$$
W''(s) \geq c_0\left|s\right|^2 \geq \left|1-c_0\right|^2 = C > 0
$$

whenever $s \in [-1+\eta, 1+c_0]$ and that in this case we also have $W(s) \leq C\left(W'(s)\right)^2$. We can thus conclude that

$$
\begin{aligned}
&\int_{\Gamma\cap\{|\varphi_\varepsilon|\geq 1-\eta\}} \frac{\varepsilon}{2}|\nabla_\Gamma\varphi_\varepsilon(p)|^2 + \varepsilon^{-1}W(\varphi_\varepsilon(p)) + \frac{1}{\varepsilon}\left(W'(\varphi_\varepsilon(p))\right)^2\, d\mathcal{H}^2(p) \\
\leq&C\int_{\Gamma\cap\{|\varphi_\varepsilon|\geq 1-\eta\}} \left(\varepsilon W''(\varphi_\varepsilon(p))\left|\nabla_\Gamma\varphi_\varepsilon(p)\right|^2 + \frac{1}{\varepsilon}(W'(\varphi_\varepsilon(p)))^2\right)\, d\mathcal{H}^2(p).
\end{aligned}
$$

Since g is either zero or linear on the interval $[-1+\eta, 1+\eta]$, it follows that $g'(s) = \mathcal{O}(\eta)$ on this interval. We can thus deduce the assertion of Lemma 8.16 from (8.40). □

Local estimates on the discrepancy measure

Since the manifold Γ is compact, it is possible to cover it with a finite atlas. Thus the first step towards the proof of Proposition 8.13 is to prove that for functions $\varphi_\varepsilon, \mu_\varepsilon$ and θ_ε fulfilling the assumptions of the proposition a certain local version of the desired estimate on the discrepancy measure holds.

We will first work under the assumption that μ_ε and θ_ε are sufficient small before turning our attention to the cases in which μ_ε and θ_ε are large. The argument requires us to carefully choose local coordinates.

As in Section 8.2.1, we start again with normal coordinates induced by the exponential map exp_p around every point $p \in \Gamma$. By the compactness of Γ, there is a real number $r > 0$ such that for every $p \in \Gamma$ the maps exp_p are diffeomorphisms from $B_r(0)$ onto the corresponding images.

Let now $R > 2$ be arbitrary. Thus we can introduce the rescaled injectivity radius $\tilde{r} := \frac{r}{R}$. Then the maps exp_p are still diffeomorphisms from $B_{\tilde{r}}(0) \subset B_{\frac{r}{2}}(0)$ onto a suitable neighborhood of $p \in \Gamma$. For $\varepsilon \leq \frac{\tilde{r}}{R} = \frac{r}{R^2}$ we can choose a finite collection of points (possibly depending on the factor R) $\{p_i \in \Gamma\}_{i=1}^{K(R)}$ such that the domains $exp_{p_i}(B_\varepsilon(0))$ cover the compact manifold. Then surely

$$\left\{ exp_{p_i}|_{B_{\tilde{r}}(0)}(B_{\tilde{r}}(0)), (exp_{p_i})^{-1}|_{B_{\tilde{r}}(0)} \right\}_{i=1}^{K(R)}$$

is an atlas of Γ which covers Γ by even larger domains and has the technical advantage that after rescaling, it will be possible to cover Γ by images of the unit ball. If we denote the metric tensor by $g_{exp}(\cdot,\cdot)$, we define $g_{exp,ij}$ to be its entries with respect to these coordinates, i.e. $g_{exp,ij} := g_{exp}(\partial_i, \partial_j)$ and let $|g_{exp}| := \det\left((g_{exp,ij})_{i,j=1}^n\right)$. Moreover, we denote the entries in the inverse $\left((g_{exp,ij})_{i,j=1}^n\right)^{-1}$ by g_{exp}^{ij}.

We now choose rescaled coordinates

$$B_{\frac{\tilde{r}}{\varepsilon}}(0) \to B_{\tilde{r}}(0), \; y \mapsto \varepsilon y.$$

For ε sufficiently small it is possible to choose $R > 2$ such that $\varepsilon = \frac{\tilde{r}}{R}$. We proceed by defining for all $y \in B_{\frac{\tilde{r}}{\varepsilon}}(0) = B_R(0)$ the functions

$$F_i(y) := \varphi_\varepsilon(exp_{p_i}(\varepsilon y))$$

and

$$M_i(y) := \varepsilon\left(\mu_\varepsilon(exp_{p_i}(\varepsilon y)) + \theta_\varepsilon(exp_{p_i}(\varepsilon y))\right)$$

where $i = 1, \ldots, K$.

The functions F_i and M_i fulfil for all $\omega \in H^1(B_R(0))$

$$\int_{B_R(0)} A(\varepsilon y)\nabla F_i(y) \cdot \nabla\omega(y) + \sqrt{|g_{exp}(\varepsilon y)|}W'(F_i(y))\omega(y)\; dy$$
$$= \int_{B_R(0)} M_i(y)\omega(y)\sqrt{|g_{exp}(y)|}\; dy$$

by virtue of (2.6) where

$$A(x) := (a_{ij}(x))_{i,j=1}^n := \sqrt{|g_{exp}|}\left(g_{exp}^{ij}\right)_{i,j=1}^n.$$

For L defined by

$$Lu(y) := -\sum_{i,j} \partial_{y_i} \left(a_{ij}(\varepsilon y) \partial_{y_j} u(\varepsilon y) \right),$$

F_i and M_i are thus weak solutions to

$$LF_i + \widetilde{W}'(F_i) = \widetilde{M}_i$$

in $B_R(0)$ where $\widetilde{W}'(y, \varphi_\varepsilon) := \sqrt{|g_{exp}(\varepsilon y)|} W'(\varphi_\varepsilon)$ and $\widetilde{M}_i(y) := M_i(y)\sqrt{|g_{exp}(\varepsilon y)|}$. For further simplification, we introduce the notation $A_\varepsilon(y) := A(\varepsilon y)$.

Observe that the manifold Γ is assumed to be smooth and compact and the functions $g_{exp,ij}$ and g_{exp}^{ij} are therefore at least locally Lipschitz. As a result, they are globally Lipschitz as well. We exploit this fact to deduce for later use the estimates

$$\begin{aligned}\|A_\varepsilon(y) - \mathrm{Id}\|_{C^0(B_R(0))} = \|A_\varepsilon(y) - A(0)\|_{C^0(B_R(0))} &\le C \sup_{y \in B_R(0)} |\varepsilon y| \le C \frac{r}{R^2} R \\ &\le CR^{-1} \end{aligned} \tag{8.41}$$

and

$$\||g_{exp}(\varepsilon y)| - 1\|_{C^0(B_R(0))} = \||g_{exp}(\varepsilon y)| - |g_{exp}(0)|\|_{C^0(B_R(0))} \le CR^{-1}. \tag{8.42}$$

The following lemma is then a first local estimate on the discrepancy measure for the rescaled functions F_i and M_i.

Lemma 8.17. For every $\eta > 0$ there exist a positive constant $R(\eta) > 2$ such that for every $R \ge R(\eta)$ and F_i, M_i weak solutions to

$$LF_i + \widetilde{W}'(F_i) = \widetilde{M}_i \tag{8.43}$$

in $B_R(0)$ as above with the additional assumption that

$$\|\widetilde{M}_i\|_{L^2(B_R(0))} \le CR^{-1}(\eta), \tag{8.44}$$

the estimate

$$\begin{aligned}&\int_{B_1} \left(A(\varepsilon x) \nabla F_i(x) \cdot \nabla F_i(x) - 2\sqrt{|g_{exp}(\varepsilon x)|} W(F_i(x)) \right)_+ \, dx \\ \le& \eta \int_{B_2} A(\varepsilon x) \nabla F_i(x) \cdot \nabla F_i(x) + \left[W'(F_i(x))^2 + W(F_i(x)) + |M_i(x)|^2 \right] \sqrt{|g_{exp}(\varepsilon x)|} \, dx \\ &+ \int_{\{x \in B_1 | |F_i| \ge 1-\eta\}} A(\varepsilon x) \nabla F_i(x) \cdot \nabla F_i(x) \, dx \end{aligned} \tag{8.45}$$

holds. Moreover, $R(\eta)$ is independent of F_i and M_i.

Proof. Let B_1^η be given by $B_1^\eta := \{x \in B_1(0) \| |F_i| \le 1-\eta\}$. Since

$$\begin{aligned}
&\int_{B_1(0)} \left(A_\varepsilon(x)\nabla F_i(x)\cdot\nabla F_i(x) - 2\widetilde{W}(F_i(x))\right)_+ \, dx \\
&= \int_{B_1^\eta} \left(A_\varepsilon(x)\nabla F_i(x)\cdot\nabla F_i(x) - 2\widetilde{W}(F_i(x))\right)_+ \, dx \\
&\quad + \int_{B_1(0)\setminus B_1^\eta} \left(A_\varepsilon(x)\nabla F_i(x)\cdot\nabla F_i(x) - 2\widetilde{W}(F_i(x))\right)_+ \, dx \\
&\le \int_{B_1^\eta} \left(A_\varepsilon(x)\nabla F_i(x)\cdot\nabla F_i(x) - 2\widetilde{W}(F_i(x))\right)_+ \, dx \\
&\quad + \int_{(B_1^\eta)^c} A_\varepsilon(x)\nabla F_i(x)\cdot\nabla F_i(x) \, dx
\end{aligned}$$

it is then sufficient to estimate the integral over B_1^η in order to prove the lemma. We distinguish the two cases

$$(1)\ \left|B_1^\eta\right| \le \eta^m \text{ and } (2)\ \left|B_1^\eta\right| > \eta^m$$

where $m := \frac{2q}{q-2}$ and $q = \frac{2n}{n-2}$ for $n > 2$ and $q = 7$ else.

Let us first consider the case $\left|B_1^\eta\right| \le \eta^m$.

Since $W(F_i(x))$ is non-negative for all $x \in \Gamma$, it is enough to estimate $A\nabla F_i(x)\cdot\nabla F_i(x)$ over B_1^η. To this end observe that by the compactness of Γ we can find an upper bound on all entries in A such that

$$\int_{B_1^\eta(0)} A_\varepsilon(x)\nabla F_i(x)\cdot\nabla F_i(x) \, dx \le C \left\|\nabla F_i\right\|^2_{L^2(B_1^\eta(0))}. \tag{8.46}$$

We thus simplify the task at hand by proving an estimate for $\left\|\nabla F_i\right\|_{L^2(B_1^\eta(0))}$.

Using Young's inequality and the Sobolev embedding $W^{1,2}(B_1(0)) \hookrightarrow L^q(B_1(0))$ we deduce

$$\left\|\nabla F_i\right\|_{L^2(B_1^\eta)} \le \left|B_1^\eta\right|^{\frac{q-2}{2q}} \left\|\nabla F_i\right\|_{L^q(B_1^\eta)} \le C\eta^{\frac{m}{m}} \left\|\nabla F_i\right\|_{W^{1,2}(B_1(0))}.$$

A standard elliptic estimate (cf. [GT01, Theorem 8.8, Theorem 8.12]) yields

$$\begin{aligned}
&\left\|\nabla F_i\right\|^2_{W^{1,2}(B_1(0))} \le C\left(\left\|LF_i\right\|^2_{L^2(B_2(0)} + \left\|\nabla F_i\right\|^2_{L^2(B_2(0))}\right) \\
&\le C\Bigg(\left\|\sqrt{|g_{exp}|}M_i\right\|^2_{L^2(B_2(0)} + \left\|\sqrt{|g_{exp}|}W'(F_i)\right\|^2_{L^2(B_2(0)} \\
&\qquad\qquad + \int_{B_2(0)} A_\varepsilon(x)\nabla F_i(x)\cdot\nabla F_i(x) \, dx\Bigg)
\end{aligned}$$

where we have used the ellipticity of A_ε and that $\sqrt{|g_{exp}|}$ is bounded from below by the compactness of Γ. Note that we only need the L^2-norm of the gradient of F_i on the right hand-side since the operator L does not contain terms of lower order, compare also [Eva10, Proof of Theorem 1, §6.3.1].

Together these estimates imply

$$\begin{aligned}
\left\|\nabla F_i\right\|^2_{L^2(B_1^\eta)} \le C\eta^2 \Bigg(&\left\|\sqrt{|g_{exp}|}M_i\right\|^2_{L^2(B_2(0))} \\
&+ \left\|\sqrt{|g_{exp}|}W'(F_i)\right\|^2_{L^2(B_2(0)} + \int_{B_2(0)} A_\varepsilon(x)\nabla F_i(x)\cdot\nabla F_i(x) \, dx\Bigg)
\end{aligned}$$

and thus we infer from (8.46)

$$\int_{B_1(0)} A_\varepsilon \nabla F_i \cdot \nabla F_i \, dx \le C\eta^2 \int_{B_2(0)} \left[|M_i|^2 + \left(W'(F_i)\right)^2 \right] \sqrt{|g_{exp}|} + A_\varepsilon \nabla F_i \cdot \nabla F_i \, dx$$
$$+ \int_{(B_1^\eta)^c} A_\varepsilon \nabla F_i \cdot \nabla F_i \, dx.$$

It remains to prove the estimate in the second case, namely if $\left|B_1^\eta\right| \ge \eta^m$. To this end, we assume that the assertion of the lemma is false and proceed by contradiction. We assume that for each $j \in \mathbb{N}$ there exist functions F_i^j and M_i^j, a ball B_j and let L_j be the local form of the Laplace-Beltrami operator with respect to the coordinates introduced above for $R = j$. We suppose in the following that for these functions F_i^j and M_i^j together with the ball B_j, the estimate (8.45) is wrong. In particular, our assumptions imply that the Matrix A_j associated with the operator L_j fulfils

$$\|A_j(y) - \mathrm{Id}\|_{C^0(B_j(0))} \le C j^{-1}, \tag{8.47}$$

in accordance with (8.41).

Let now $\kappa > 0$ and ζ be a smooth cut-off function with $0 \le \zeta \le 1$ such that $\zeta \equiv 1$ on $B_\kappa(0)$ and $\zeta \equiv 0$ outside of $B_{2\kappa}(0)$. Moreover, let $k = \frac{2q}{q-2}$. After multiplying with $\zeta^k F_i^j$ and integrating over B_j, equation (8.43) reads

$$0 = \int_{B_j} \left(A_j \nabla F_i^j\right) \cdot \nabla \left(\zeta^k F_i^j\right) + \widetilde{W}'(F_i^j)\zeta^k F_i^j - \widetilde{M}_i^j \zeta^k F_i^j \, dx$$

or, equivalently,

$$\int_{B_j} \zeta^k \left(A_j \nabla F_i^j\right) \cdot \nabla F_i^j + \widetilde{W}'(F_i^j)\zeta^k F_i^j \, dx$$
$$= \int_{B_j} -k\zeta^{k-1} F_i^j \left(A_j \nabla F_i^j\right) \cdot \nabla \zeta + \widetilde{M}_i^j \zeta^k F_i^j \, dx \tag{8.48}$$

The summand $\int_{B_j} k\zeta^{k-1} F_i^j \left(A_j \nabla F_i^j\right) \cdot \nabla \zeta \, dx$ can be estimated with the help of Young's inequality. Choosing $\tilde{q} = \frac{q}{2}$ and $\tilde{q}' = \frac{q}{q-2}$, we thereby obtain

$$\begin{aligned}
&\int_{B_j} k\zeta^{k-1} F_i^j \left(A_j \nabla F_i^j\right) \cdot \nabla \zeta \, dx \\
&= \int_{B_j} \left(\zeta^{k/2} (A_j)^{1/2} \nabla F_i^j\right) \cdot \left(k\zeta^{k/2-1} F_i^j (A_j)^{1/2} \nabla \zeta\right) \, dx \\
&\ge -\frac{1}{2} \int_{B_j} \left[\zeta^k \left(A_j \nabla F_i^j\right) \cdot \nabla F_i^j + k^2 \zeta^{k-2} (F_i^j)^2 \left(A_j \nabla \zeta\right) \cdot \nabla \zeta \right] \, dx \\
&\ge -\frac{1}{2} \int_{B_j} \left[\zeta^k \left(A_j \nabla F_i^j\right) \cdot \nabla F_i^j + \delta \left(k\zeta^{k-2} (F_i^j)^2\right)^{\tilde{q}} + C_\delta \left(k (A_j \nabla \zeta) \cdot \nabla \zeta\right)^{\tilde{q}'} \right] \, dx.
\end{aligned}$$

In the same way, we can estimate the integral $\int_{B_j} \widetilde{M}_i^j \zeta^k F_i^j \, dx$ if we use Young's inequality to deduce

$$\begin{aligned}\int_{B_j} \widetilde{M}_i^j \zeta^k F_i^j \, dx &\le \frac{1}{2}\int_{B_j} \zeta^k (\widetilde{M}_i^j)^2 + \zeta^k (F_i^j)^2 \, dx \\ &= \frac{1}{2}\int_{B_j} \zeta^k (\widetilde{M}_i^j)^2 + \zeta^{k-2} (F_i^j)^2 \zeta^2 \, dx \\ &\le \frac{1}{2}\left[\int_{B_j} (\widetilde{M}_i^j)^2 \, dx + \delta \int_{B_j} \zeta^k (F_i^j)^q \, dx + C_\delta \int_{B_j} \zeta^k \, dx\right],\end{aligned}$$

where we have chosen the pair $\tilde{q}$ and $\tilde{q}'$ from above as the exponents in the second application of Young's inequality. Now recall that $rW'(r) \ge c_1 |r|^q - c_2$ and choose $\delta = \frac{c_1}{2k^{q/2}}$. Using the last two inequalities, (8.48) becomes

$$\int_{B_j} \zeta^k \left((A_j \nabla F_i^j) \cdot \nabla F_i^j + \left|F_i^j\right|^q \right) \le C\left(c_1, c_2, q, \left\|\widetilde{M}_i^j\right\|_{L^2(B_j)}, \|\nabla \zeta\|_{L^k(B_j)}\right).$$

Since the A_j are uniformly elliptic and since the sequence $\{\widetilde{M}_i^j\}_{j\in\mathbb{N}} \subset L^2(B_r(0) \cap B_j)$ is bounded by assumption, this estimate yields for any $\kappa > 0$

$$\left\|F_i^j\right\|_{W^{1,2}(B_\kappa(0)\cap B_j)} \le C = C(\kappa).$$

Using Sobolev embeddings, we hence find that $W'(F_i^j)$ is bounded in $L^2(B_\kappa(0) \cap B_j)$ and by elliptic theory (see again [GT01]) we deduce for any $\kappa' < \kappa$

$$\left\|F_i^j\right\|_{W^{2,2}(B'_\kappa(0)\cap B_j)} + \left\|W'(F_i^j)\right\|_{L^2(B'_\kappa(0)\cap B_j)} \le C = C(\kappa).$$

Since κ was arbitrary, we can write κ instead of κ' in the estimate above.

We are now interested in the limit behavior of the tuple (F_i^j, M_i^j) as $j \to \infty$. By the previous estimates and the general assumptions in the statement of the lemma, we can deduce the existence of a subsequence $\{j_k\}_{k\in\mathbb{N}}$ with $j_k \to \infty$ such that the following convergences hold for any $\kappa > 0$:

(i) $\widetilde{M}_i^{j_k} \to 0$ in $L^2(B_\kappa(0))$ by (8.44).

(ii) $F_i^{j_k} \to F$ in $W^{1,2}(B_\kappa(0))$ and for $0 < \alpha < \frac{1}{2}$ in $C^{0,\alpha}(B_\kappa(0))$ for some $F \in W^{2,2}_{loc}(\mathbb{R}^n)$ by the compact Sobolev embedding $W^{2,2}(B_\kappa(0)) \hookrightarrow W^{1,2}(B_\kappa(0))$ or by the compact embedding $W^{2,2}(B_\kappa(0)) \hookrightarrow C^{0,\alpha}(B_\kappa(0))$ respectively.

(iii) $W'(F_i^{j_k}) \to W'(F)$ in $L^q(B_\kappa(0))$ for $q \in [1,2)$.

(iv) $W(F_i^{j_k}) \to W(F)$ in $L^1(B_\kappa(0))$.

By the dominated convergence theorem and estimate (8.42), (iii) implies

$$\int_{B_\kappa(0)} \sqrt{|g_{exp}|} W'(F_i^{j_k}) \omega \, dx \to \int_{B_\kappa(0)} W'(F) \omega \, dx.$$

At the same time, estimate (8.47) yields

$$
\begin{aligned}
&\left|\int_{B_\kappa(0)} A_{j_k}(x)\nabla F_i^{j_k}(x)\cdot\nabla\omega(x) - \nabla F(x)\cdot\nabla\omega(x)\ dx\right| \\
&\le \int_{B_\kappa(0)} \left|A_{j_k}(x)\nabla F_i^{j_k}(x)\cdot\nabla\omega(x) - \nabla F_i^{j_k}(x)\cdot\nabla\omega(x)\right| + \left|\nabla F_i^{j_k}(x)\cdot\nabla\omega(x) - \nabla F(x)\cdot\nabla\omega(x)\right|\ dx \\
&\le \|A_{j_k} - \mathrm{Id}\|_{L^\infty(B_\kappa(0))}\int_{B_\kappa(0)} \nabla F_i^{j_k}(x)\cdot\nabla\omega(x)\ dx + \int_{B_\kappa(0)} \left|\nabla F_i^{j_k}(x) - \nabla F(x)\right||\nabla\omega(x)|\ dx
\end{aligned}
$$

and thus

$$
\left|\int_{B_\kappa(0)} A_{j_k}(x)\nabla F_i^{j_k}(x)\cdot\nabla\omega(x) - \nabla F(x)\cdot\nabla\omega(x)\ dx\right| \to 0
$$

by the convergence of $F_i^{j_k}$ in (ii).

Given that $F_i^{j_k}$ and $\widetilde{M}_i^{j_k}$ fulfil

$$
\begin{aligned}
\int_{B_\kappa(0)} A_{j_k}(x)\nabla F_i^{j_k}(x)\cdot\nabla\omega(x)\ dx + \int_{B_\kappa(0)} \widetilde{W}'(F_i^{j_k}(x))\omega(x)\ dx& \\
= \int_{B_\kappa(0)} M_i^{j_k}(x)\omega(x)\ dx&
\end{aligned}
$$

for each test function $\omega \in W^{1,2}(B_\kappa(0))$, the above convergences together with (i) imply that the limit function F fulfils

$$
-\Delta F + W'(F) = 0
$$

weakly in $W^{1,2}(B_\kappa(0))$ for all $\kappa > 0$. As a result, we can apply Lemma 8.14 which gives

$$
\lim_{k\to\infty}\int_{B_1(0)} \left(A_{j_k}\nabla F_i^{j_k}\cdot\nabla F_i^{j_k} - 2W(F_i^{j_k})\right)_+ = \int_{B_1(0)} \left(|\nabla F|^2 - 2W(F)\right)_+ = 0
$$

Hence the left hand side in (8.45) is non-positive in the limit $k\to\infty$. On the other hand, we assumed

$$
\left|\left\{x\in B_1(0) : \left|F_i^{j_k}\right| \le 1-\eta\right\}\right| \ge \eta^m.
$$

and since $F_i^{j_k} \to F$ a.e. this also implies

$$
|\{x\in B_1(0) : |F| \le 1-\eta\}| \ge \eta^m.
$$

By the convergence results in (i)–(iv) we hence find

$$
\begin{aligned}
\lim_{k\to\infty}\eta\int_{B_1(0)} A_{j_k}\nabla F_i^{j_k}\cdot\nabla F_i^{j_k} + \widetilde{W}(F_i^{j_k})\ dx &= \eta\int_{B_1(0)} |\nabla F|^2 + W(F)\ dx \\
\ge \eta\int_{\{x\in B_1(0)||F|\le 1-\eta\}} W(F) &\ge \eta\eta^m \min_{s\in[-1+\eta,1-\eta]} W(s)
\end{aligned}
$$

and thus the right hand side of (8.45) is uniformly positive in k, in contradiction to the assumption that the converse is true. Thus the assertion of the lemma is proved. □

Proof of Proposition 8.13

Using Lemma 8.16 and 8.17 we can now proceed with the proof of Proposition 8.13.

As before, we consider again normal coordinates induced by a suitable rescaling of the exponential maps exp_p around every point $p \in \Gamma$. Again we denote by r the injectivity radius, which is uniform on Γ since Γ is compact. Let η be any fixed small positive constant. Let $R(\eta) > 2$ be the constant from Lemma 8.17 such that estimate (8.45) holds. For ε small enough choose $R > R(\eta)$ such that $\varepsilon = \frac{r}{R^2}$. With the same construction as before, we obtain again an atlas for the manifold Γ that scales with ε. We denote it by

$$\left\{ exp^{\varepsilon}_{p_i}\big|_{B_R(0)}(B_R(0)), (exp^{\varepsilon}_{p_i})^{-1}\big|_{B_R(0)} \right\}_{i=1}^{K(r)}.$$

Remark 8.18. Note that $B_2(0) \subset B_R(0)$ and that the points p_i where chosen in such a way that the rescaled atlas covers Γ even if one restricts the charts to $B_1(0) \subset B_R(0)$. Moreover, we point out that by the covering theorem [Fed69, Theorem 2.8.14] and Remark 2.4.8 therein for each $p \in \Gamma$ the number

$$\#\left\{ i \in 1, \dots, K(r) \,\middle|\, p \in exp^{\varepsilon}_{p_i}\big|_{B_R(0)}(B_R(0)) \right\}$$

is bounded by some constant $C(\Gamma)$ which only depends on the dimension of Γ and is in particular independent of ε.

With respect to this atlas, the localized functions F_i, M_i defined as before fulfil

$$\begin{aligned}\int_{B_R(0)} A(\varepsilon y)\nabla F_i(y)\cdot\nabla\omega(y) + \sqrt{|g_{exp}(\varepsilon y)|}W'(F_i(y))\omega(y)\;dy&\\ = \int_{B_R(0)} M_i(y)\omega(y)\sqrt{|g_{exp}(y)|}\;dy&\end{aligned}$$

for all $\omega \in H^1(B_R(0))$ and are thus weak solutions to

$$LF_i + \widetilde{W}'(F_i) = \widetilde{M_i}$$

in $B_R(0)$.

Restricting ourselves for the moment to the set of all $i \in \mathcal{I}_1 \subset \{1 \dots K\}$ given by

$$\mathcal{I}_1 := \left\{ i \in \{1\dots K\} \,\middle|\, \left\| \mu_\varepsilon + \frac{1}{2}\theta_\varepsilon \right\|_{L^2(exp_{p_i}(B_{\tilde r}(0))\cap\Gamma)} \le \varepsilon^{\frac{n}{2}-1}R^{-1} \right\} \tag{8.49}$$

we can thus apply Lemma 8.17 since (8.49) implies $\|\widetilde{M_i}\|_{L^2(B_R(0))} \le R^{-1}$.

Hence we have

$$\begin{aligned}&\int_{B_1}\left(A\nabla F_i(x)\cdot\nabla F_i(x) - 2\sqrt{|g_{exp}|}W(F_i(x))\right)_+\;dx\\ \le&\eta\int_{B_2} A\nabla F_i(x)\cdot\nabla F_i(x) + \left[W'(F_i(x))^2 + W(F_i(x)) + |M_i(x)|^2\right]\sqrt{|g_{exp}|}\;dx\\ &+\int_{\{x\in B_1 \| \varphi|\ge 1-\eta\}} A\nabla F_i(x)\cdot\nabla F_i(x)\;dx.\end{aligned}$$

Keeping in mind that by the compactness of Γ both $|g_{exp}|$ and $|g_{exp}|^{-1}$ are bounded, transferring back to $\varphi_\varepsilon, \mu_\varepsilon$ and θ_ε on Γ leads to

$$\begin{aligned}
&\int_{exp^\varepsilon_{p_i}(B_1(0))} (\xi^\varepsilon(\varphi_\varepsilon))_+ \; d\mathcal{H}^2 \\
\leq &\eta C \int_{exp^\varepsilon_{p_i}(B_2(0))} \left(\frac{\varepsilon}{2} |\nabla_\Gamma \varphi_\varepsilon|^2 + \varepsilon^{-1} W(\varphi_\varepsilon) + \frac{1}{\varepsilon} W'(\varphi_\varepsilon)^2 + \varepsilon (|\mu_\varepsilon|^2 + \frac{1}{2} |\theta_\varepsilon|^2) \right) \; d\mathcal{H}^2 \\
&+ C \int_{exp^\varepsilon_{p_i}(B_1(0)) \cap \{p \in \Gamma || \varphi_\varepsilon| \geq 1 - \eta\}} \varepsilon |\nabla_\Gamma \varphi_\varepsilon|^2 \, \mathrm{d}\mathcal{H}^2.
\end{aligned}$$

Taking the sum over all $i \in \mathcal{I}_1$ yields

$$\begin{aligned}
&\sum_{i \in \mathcal{I}_1} \int_{exp^\varepsilon_{p_i}(B_1(0))} (\xi^\varepsilon(\varphi_\varepsilon))_+ \; d\mathcal{H}^2 \\
&\leq C(\Gamma) \eta \int_\Gamma \frac{\varepsilon}{2} |\nabla_\Gamma \varphi_\varepsilon|^2 + \varepsilon^{-1} W(\varphi_\varepsilon) + \frac{1}{\varepsilon} W'(\varphi_\varepsilon)^2 + \varepsilon (|\mu_\varepsilon|^2 + \frac{1}{2} |\theta_\varepsilon|^2) \; d\mathcal{H}^2 \\
&+ C(\Gamma) \int_{\{p \in \Gamma || \varphi_\varepsilon| \geq 1 - \eta\}} \varepsilon |\nabla_\Gamma \varphi_\varepsilon|^2 \, \mathrm{d}\mathcal{H}^2,
\end{aligned}$$

where the constant $C(\Gamma)$ incorporates the constant from Remark 8.18 in order to account for the possible overlap between the regions $exp^\varepsilon_{p_i}(B_1(0))$.

Finally, using the estimates away from the interface in Lemma 8.16 provides

$$\begin{aligned}
&\sum_{i \in \mathcal{I}_1} \int_{exp^\varepsilon_{p_i}(B_1(0))} (\xi^\varepsilon(\varphi_\varepsilon))_+ \; d\mathcal{H}^2 \\
&\leq C\eta \int_\Gamma \frac{\varepsilon}{2} |\nabla_\Gamma \varphi_\varepsilon|^2 + \varepsilon^{-1} W(\varphi_\varepsilon) \; d\mathcal{H}^2 + C\varepsilon \int_\Gamma (|\mu_\varepsilon|^2 + \frac{1}{2} |\theta_\varepsilon|^2) \; d\mathcal{H}^2
\end{aligned} \tag{8.50}$$

if one takes into account that $(W'(r))^2 \leq C W(r)$ whenever $|r| \leq 1$.

In order to complete the proof, let us now denote $\mathcal{I}_2 := \{1, \ldots, K\} \setminus \mathcal{I}_1$ and introduce

$$G^{-1} = \left(g^{\varepsilon, ij}_{\mathrm{exp}} \right)_{i,j \in \{1,2\}}.$$

We observe that by interior regularity results for elliptic equations (compare [GT01, Theorem 8.8, Theorem 8.12]) we have that

$$\begin{aligned}
\int_{B_1(0)} \nabla F_i \cdot G^{-1} \nabla F_i \sqrt{|g^\varepsilon_{\mathrm{exp}}|} \; dx &\leq C \int_{B_2(0)} \left(W'(F_i)^2 + |M_i|^2 + |F_i|^2 \right) \sqrt{|g^\varepsilon_{exp}|} \; dx \\
&\leq C |B_1(0)| + C \int_{B_2(0)} \left(|M_i|^2 + W'(F_i)^2 \chi_{\{|F_i| \geq 1\}} \right) \sqrt{|g^\varepsilon_{exp}|} \; dx
\end{aligned}$$

since the F_i solve

$$L F_i + \widetilde{W}'(F_i) = \widetilde{M}_i$$

and $W'(r)$ is bounded for $|r| \leq 1$.

As before, transferring back to $\varphi_\varepsilon, \mu_\varepsilon$ and θ_ε on Γ and the fact that Γ is compact allows us to deduce

$$\begin{aligned}\int_{exp^\varepsilon_{p_i}(B_1(0))} \varepsilon \left|\nabla_\Gamma \varphi_\varepsilon\right|^2 \leq & C\varepsilon \int_{\exp^\varepsilon_{p_i}(B_1(0))} \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 \, d\mathcal{H}^2 \\ & + C\varepsilon^{-1} \int_{\{\exp_{p_i}(B_1(0)) \| \varphi_\varepsilon| \geq 1\}} W'(\varphi_\varepsilon)^2 \, d\mathcal{H}^2 \\ & + C\varepsilon^{-1} \left|\exp^\varepsilon_{p_i}(B_1(0))\right|.\end{aligned}$$

Again, we take the sum over all $i \in \mathcal{I}_2$ and allow for a constant $C(\Gamma)$ due to possible overlap, see Remark 8.18. This procedure implies

$$\begin{aligned}\sum_{i\in\mathcal{I}_2} \int_{exp^\varepsilon_{p_i}(B_1(0))} \varepsilon \left|\nabla_\Gamma \varphi_\varepsilon\right|^2 \leq & C(\Gamma)\varepsilon \int_\Gamma \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 \, d\mathcal{H}^2 \\ & + C(\Gamma)\varepsilon^{-1} \int_{\{p\in\Gamma \| \varphi_\varepsilon| \geq 1\}} W'(\varphi_\varepsilon)^2 \, d\mathcal{H}^2 \\ & + C\varepsilon^{-1} \sum_{i\in\mathcal{I}_2} \left|exp^\varepsilon_{p_i}(B_1(0))\right|.\end{aligned}$$

Similar to the proof of Lemma 8.16, we multiply (2.5) by $W'(\varphi_\varepsilon)$ and integrate over Γ to deduce

$$\int_\Gamma \varepsilon W''(\varphi_\varepsilon) \left|\nabla_\Gamma \varphi_\varepsilon\right|^2 + \frac{1}{\varepsilon}(W'(\varphi_\varepsilon))^2 \, d\mathcal{H}^2 \leq \int_\Gamma \frac{\varepsilon}{2}\left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 + \frac{1}{2\varepsilon}(W'(\varphi_\varepsilon))^2 \, d\mathcal{H}^2.$$

For $|s| \geq 1$ we have $W''(s) > 0$ and thus we infer

$$\varepsilon^{-1} \int_{\{p\in\Gamma \| \varphi_\varepsilon| \geq 1\}} W'(\varphi_\varepsilon)^2 \, d\mathcal{H}^2 \leq \int_\Gamma \frac{\varepsilon}{2}\left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 \, d\mathcal{H}^2.$$

As a direct consequence, we obtain

$$\sum_{i\in\mathcal{I}_2} \int_{exp^\varepsilon_{p_i}(B_1(0))} \varepsilon \left|\nabla_\Gamma \varphi_\varepsilon\right|^2 \leq C(\Gamma)\varepsilon \int_\Gamma \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 \, d\mathcal{H}^2 + C\varepsilon^{-1} \sum_{i\in\mathcal{I}_2} \left|exp^\varepsilon_{p_i}(B_1(0))\right|. \tag{8.51}$$

In order to derive an estimate for the constant $\sum_{i\in\mathcal{I}_2} \left|exp^\varepsilon_{p_i}(B_1(0))\right|$ observe that for all $i \in \mathcal{I}_2$ in question

$$\int_{exp^\varepsilon_{p_i}(B_1(0))} \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 \, d\mathcal{H}^2 \geq R^{-2} \geq \varepsilon^{-2} R^{-2} \frac{\left|exp^\varepsilon_{p_i}(B_1(0))\right|}{|B_1|}$$

where $B_1 \subset \Gamma$ denotes the unit Ball in Γ. Remark 8.18 ensures again that a possible overlap between the domains $exp^\varepsilon_{p_i}(B_1(0))$ can be controlled independently of ε. Hence we deduce

$$\begin{aligned}\sum_{i\in\mathcal{I}_2} \left|exp^\varepsilon_{p_i}(B_1(0))\right| &\leq \varepsilon^2 \left|B_1\right| R^2 \sum_{i\in\mathcal{I}_2} \int_{exp^\varepsilon_{p_i}(B_1(0))} \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 \, d\mathcal{H}^2 \\ &\leq \varepsilon^2 \left|B_1\right| R^2 C(\Gamma) \int_\Gamma \left(\mu_\varepsilon + \frac{1}{2}\theta_\varepsilon\right)^2 \, d\mathcal{H}^2.\end{aligned}$$

Thus estimate (8.51) reads

$$\sum_{i\in\mathcal{I}_2}\int_{exp^{\varepsilon}_{p_i}(B_1(0))}\varepsilon\,|\nabla_\Gamma\varphi_\varepsilon|^2\leq C\left[1+R^2\right]\varepsilon\int_\Gamma\left(\mu_\varepsilon+\frac{1}{2}\theta_\varepsilon\right)^2\,d\mathcal{H}^2. \tag{8.52}$$

Since the family

$$\{exp^{\varepsilon}_{p_i}(B_1(0))\}_{i=1}^{K(r)}$$

of domains covers Γ thanks to the construction of the original atlas, the estimates (8.50) and (8.52) yield

$$\begin{aligned}&\int_\Gamma(\xi^\varepsilon(\varphi_\varepsilon))_+\,\,d\mathcal{H}^2\\&\leq C\eta\int_\Gamma\frac{\varepsilon}{2}|\nabla_\Gamma\varphi_\varepsilon|^2+\varepsilon^{-1}W(\varphi_\varepsilon)\,\,d\mathcal{H}^2+C\varepsilon M(\eta)\int_\Gamma(|\mu_\varepsilon|^2+\frac{1}{2}\,|\theta_\varepsilon|^2)\,\,d\mathcal{H}^2\end{aligned}$$

which completes the proof. Note that M has to depend on η since our choice of R depends on η.

8.2.4 Proof of Proposition 8.4

The convergences of $\varphi_{\varepsilon_k},\mu_{\varepsilon_k},\theta_{\varepsilon_k},v_{\varepsilon_k}$ and u_{ε_k} follow from Proposition 8.9 for φ_{ε_k}, from Lemma 8.11 and the weak compactness of bounded sets in reflexive Banach spaces in the case of μ_{ε_k} and θ_{ε_k} and finally directly from the energy estimate for u_{ε_k}.

8.2.5 Proof of Theorem 8.5

The main part of the proof is the construction of a suitable varifold V that fulfils (8.16). To this end, we first consider the two measures λ^{ε_k} and h^{ε_k} defined by

$$\begin{aligned}\lambda^{\varepsilon_k}:&=\left[\varepsilon_k\frac{|\nabla_\Gamma\varphi_{\varepsilon_k}|^2}{2}+\frac{1}{\varepsilon_k}W(\varphi_{\varepsilon_k})\right]\,d\mathcal{H}^2(p)\,\,dt\quad\text{and}\\h^{\varepsilon_k}:&=\left[\varepsilon_k\nabla_\Gamma\varphi_{\varepsilon_k}\otimes\nabla_\Gamma\varphi_{\varepsilon_k}\right]\,\,d\mathcal{H}^2(p)\,\,dt.\end{aligned}$$

Observe that λ^{ε_k} and h^{ε_k} are bounded by the energy estimate. Thus we can use the compactness properties of Radon measures (see e.g. Theorem 3.24) to deduce the existence of measures λ and h such that

$$\begin{aligned}\left[\varepsilon_k\frac{|\nabla_\Gamma\varphi_{\varepsilon_k}|^2}{2}+\frac{1}{\varepsilon_k}W(\varphi_{\varepsilon_k})\right]d\mathcal{H}^2(p)\,\,dt&\to d\lambda(p,t)\quad\text{and}\\\left[\varepsilon_k\nabla_\Gamma\varphi_{\varepsilon_k}\otimes\nabla_\Gamma\varphi_{\varepsilon_k}\right]\,d\mathcal{H}^2(p)\,\,dt&\to dh(p,t)\end{aligned}\tag{8.53}$$

in the sense of measures.

Moreover, for any $Y,Z\in C([0,T]\times\Gamma;T\Gamma)$ the inequality

$$\begin{aligned}&\int_0^T\int_\Gamma Y^T\left(\varepsilon_k\nabla_\Gamma\varphi_{\varepsilon_k}\otimes\nabla_\Gamma\varphi_{\varepsilon_k}\right)Z\,\,d\mathcal{H}^2\,\,dt\\&\leq\int_0^T\int_\Gamma|Y||Z|\left(\frac{\varepsilon_k}{2}\,|\nabla_\Gamma\varphi_{\varepsilon_k}|^2+\frac{1}{\varepsilon_k}W(\varphi_{\varepsilon_k})\right)\,\,d\mathcal{H}^2\,\,dt\\&\quad+\int_0^T\int_\Gamma|Y||Z|\left(\frac{\varepsilon_k}{2}\,|\nabla_\Gamma\varphi_{\varepsilon_k}|^2-\frac{1}{\varepsilon_k}W(\varphi_{\varepsilon_k})\right)\,\,d\mathcal{H}^2\,\,dt\end{aligned}$$

holds.

By Proposition 8.13 the second term on the right hand side is non-positive in the limit $k \to \infty$. Thus we deduce

$$\int_0^T \int_\Gamma Y^T(dh)Z \le \int_0^T \int_\Gamma |Y||Z|\, d\lambda, \tag{8.54}$$

which proves that the measure h is absolutely continuous with respect to λ. Hence the Radon-Nikodym theorem 3.25 grants the existence of a section ω in $T^*\Gamma \otimes T^*\Gamma$ such that

$$dh(p,t) = \omega d\lambda(p,t). \tag{8.55}$$

Next we choose an index set $\mathcal{I}$ and disjoint sets $A_l \subset \Gamma, l \in \mathcal{I}$ such that $\Gamma = \dot{\bigcup}_{l\in\mathcal{I}} A_l$. Moreover, let $(U_l, \alpha_l)_{l\in\mathcal{I}}$ be an atlas of Γ such that $A_l \subset U_l$ for each $l \in \mathcal{I}$.

With respect to the charts α_l, we define local measures $h_l^{\varepsilon_k}$ on $\alpha_j(A_l) \subset \mathbb{R}^2$ by setting

$$h_l^{\varepsilon_k} := \varepsilon_k \left(g^{si} g^{rj} \frac{\partial \varphi_{\varepsilon_k}}{\partial x_r} \frac{\partial \varphi_{\varepsilon_k}}{\partial x_s} \right)_{i,j=1,2} \sqrt{|g|}\ dx\ dt.$$

Note that we use Einstein summation convention with respect to s and r and that the measures $h_l^{\varepsilon_k}$ are just $h^{\varepsilon_k} \llcorner A_l$ expressed in local coordinates. As such, the bound on h^{ε_k} from the energy estimate carries over to $h_l^{\varepsilon_k}$ and we deduce the existence of measures h_l such that

$$\varepsilon_k \left(g^{si} g^{rj} \frac{\partial \varphi_{\varepsilon_k}}{\partial x_r} \frac{\partial \varphi_{\varepsilon_k}}{\partial x_s} \right)_{i,j=1,2} \sqrt{|g|}\ dx\ dt \to dh_l(x,t) \tag{8.56}$$

in the sense of measures. We also introduce the measures $h_l^{i,j}$ as the limit measures for every entry in $h_l^{\varepsilon_k}$.

Analogously, we define the measures $\lambda_l^{\varepsilon_k}$ and λ_l as $\lambda^{\varepsilon_k} \llcorner A_l$ in local coordinates and its limit measure respectively.

Furthermore, the arguments leading to (8.54) also imply the absolute continuity $h_l << \lambda_l$. This implies the existence of λ_l-measurable functions $\nu_{i,j}$ such that

$$dh_l^{i,j}(x,t) = \nu_l^{i,j} d\lambda_l(x,t). \tag{8.57}$$

At the same time, the matrix $\left(\nu_l^{i,j}\right)_{ij}$ is symmetric and positive definite by definition and can thus be written as

$$\left(\nu_l^{i,j}\right)_{i,j=1,2} = \sum_{k=1}^2 \tilde{c}_k^l \vec{\nu}_k^l \otimes \vec{\nu}_k^l \quad \lambda_l\text{-almost everywhere.} \tag{8.58}$$

Here the $\{\vec{\nu}_k^l\}$ are an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors. The functions $\tilde{c}_k^l$ fulfil $\tilde{c}_k^l \in [0,1]$ since equation (8.54) directly shows that the matrix $(\nu_{i,j})_{ij}$ cannot have eigenvalues larger than 1. Moreover, we note for later use that

$$\sum_{k=1}^2 \vec{\nu}_k^l \otimes \vec{\nu}_k^l = \mathrm{Id}\,. \tag{8.59}$$

Let $Y \in C^1(\Gamma, T\Gamma)$ be a vector field on Γ. To simplify the following calculations, we denote the entries of the differential $\nabla_\Gamma Y$ in local coordinates by $d_{i,j}^Y$ for $i,j = 1,2$.

Now observe that the transformation formula infers for all $l \in \mathcal{I}$ and all $Y \in C^1(\Gamma, T\Gamma)$

$$\int_0^T \int_{A_l} \nabla_\Gamma Y : [\varepsilon_k \nabla_\Gamma \varphi_{\varepsilon_k} \otimes \nabla_\Gamma \varphi_{\varepsilon_k}] \ d\mathcal{H}^2(p) \ dt \\ = \int_0^T \int_{\alpha_l(A_l)} \varepsilon_k \left(d^Y_{i,j}\right)_{i,j=1,2} : \left(g^{si} g^{rj} \frac{\partial \varphi_{\varepsilon_k}}{\partial x_r} \frac{\partial \varphi_{\varepsilon_k}}{\partial x_s}\right)_{i,j=1,2} \sqrt{|g|} \ dx \ dt.$$

For the left hand-side we have

$$\int_0^T \int_{A_l} \nabla_\Gamma Y : [\varepsilon_k \nabla_\Gamma \varphi_{\varepsilon_k} \otimes \nabla_\Gamma \varphi_{\varepsilon_k}] \ d\mathcal{H}^2(p) \ dt \to \int_0^T \int_{A_l} \nabla_\Gamma Y : \omega d\lambda$$

by (8.53) and (8.55). Furthermore, the right hand-side fulfils

$$\int_0^T \int_{\alpha_l(A_l)} \varepsilon_k \left(d^Y_{i,j}\right)_{i,j=1,2} : \left(g^{si} g^{rj} \frac{\partial \varphi_{\varepsilon_k}}{\partial x_r} \frac{\partial \varphi_{\varepsilon_k}}{\partial x_s}\right)_{i,j=1,2} \sqrt{|g|} \ dx \ dt \\ \longrightarrow \int_0^T \int_{\alpha_l(A_l)} \left(d^Y_{i,j}\right)_{i,j=1,2} : \left(\nu_l^{i,j}\right)_{i,j=1,2} d\lambda_l$$

by (8.56) and (8.57). Therefore we deduce

$$\int_0^T \int_{A_l} \nabla_\Gamma Y : \omega d\lambda = \int_0^T \int_{\alpha_l(A_l)} \left(d^Y_{i,j}\right)_{i,j=1,2} : \left(\nu_l^{i,j}\right)_{i,j=1,2} d\lambda_l. \tag{8.60}$$

We now define the varifold V as follows. We return to the functions $\tilde{c}^l_k$ in (8.58) and define

$$c^l_k(x,t) := 1 + \tilde{c}^l_k(x,t) - \sum_{m=1}^{2} \tilde{c}^l_m(x,t) \tag{8.61}$$

on $\alpha_l(A_l) \times [0,T]$. The Radon measure V^l on $[0,T] \times G(A_l)$ defined by

$$dV^l_t(p,S) = \sum_{k=1}^{2} \alpha_l^* c^l_k(p,t) d\lambda(p,t) \delta_{\alpha_l^* \tilde{\nu}^l_k(p,t)}(S)$$

is a varifold for almost all times $t \in [0,T]$ since by [AFP00, Theorem 2.28] the measures λ_l and $h_l^{i,j}$ can be split into a spatial and a time part, i.e. there exist measures λ^t_l and $h_l^{i,j,t}$ such that $d\lambda = d\lambda^t_l \ dt$ and $dh_l^{i,j} = dh_l^{i,j,t} \ dt$.

Finally, we define V by

$$\int_0^T \int_{G(\Gamma)} \eta(p,S) \ dV_t(p,S) := \sum_{l \in \mathcal{I}} \int_0^T \int_{G(A_l)} \eta(p,S) \ dV^l_t(p,S) \tag{8.62}$$

for all $\eta \in C_0(G(\Gamma))$.

To conclude the proof, we show that the varifold V from (8.62) fulfils equation (8.16). Let Y be any vector field $Y \in C^1(\Gamma, T\Gamma)$. For all such vector fields Y, the convergence results from

Proposition 8.4 allow us to take the limit $k \to \infty$ in equation (8.36) and as a result, we deduce

$$\begin{aligned}
&\int_0^T \int_\Gamma 2\chi_{Q_t} \operatorname{div}_\Gamma(\mu Y)\ d\mathcal{H}^2\ dt = -\int_0^T \int_\Gamma \nabla_\Gamma Y : (\mathrm{Id} - \omega)\, d\lambda \\
&= -\int_0^T \int_\Gamma \nabla_\Gamma Y : \mathrm{Id}\ d\lambda + \sum_{l\in\mathcal{I}} \int_0^T \int_{A_l} \nabla_\Gamma Y : \omega\ d\lambda \\
&= -\int_0^T \int_\Gamma \nabla_\Gamma Y : \mathrm{Id}\ d\lambda + \sum_{l\in\mathcal{I}} \int_0^T \int_{\alpha_l(A_l)} \left(d^Y_{i,j}\right)_{i,j=1,2} : \left(\nu_l^{i,j}\right)_{i,j=1,2} d\lambda_l \\
&= -\int_0^T \int_\Gamma \nabla_\Gamma Y : \mathrm{Id}\ d\lambda + \sum_{l\in\mathcal{I}} \sum_{k=1}^2 \int_0^T \int_{\alpha_l(A_l)} \left(d^Y_{i,j}\right)_{i,j=1,2} : \left(\tilde{c}^l_k \vec{\nu}^l_k \otimes \vec{\nu}^l_k\right) d\lambda_l
\end{aligned} \tag{8.63}$$

from (8.60) and (8.58).

At the same time, we have

$$\begin{aligned}
\langle \delta V_t, Y\rangle &= \int_{G(\Gamma)} \nabla_\Gamma Y(p) : (\mathrm{Id} - S\otimes S)\, dV_t(p,S) \\
&= \sum_{l\in\mathcal{I}} \int_{G(A_l)} \nabla_\Gamma Y(p) : (\mathrm{Id} - S\otimes S)\, dV_t(p,S) \\
&= \sum_{l\in\mathcal{I}} \sum_{k=1}^2 \int_{G(A_l)} \nabla_\Gamma Y(p) : (\mathrm{Id} - S\otimes S)\, \alpha_l^* c^l_k(p,t)\ d\lambda(p,t) \delta_{\alpha_l^* \vec{\nu}^l_k}(S) \\
&= \sum_{l\in\mathcal{I}} \sum_{k=1}^2 \int_{A_l} \nabla_\Gamma Y(p) : \left(\mathrm{Id} - \alpha_l^* \vec{\nu}^l_k \otimes \alpha_l^* \vec{\nu}^l_k\right) \alpha_l^* c^l_k(p,t)\ d\lambda(p,t)
\end{aligned} \tag{8.64}$$

by the definition of the first variation of the varifold V and (8.62). To simplify the presentation, we now consider the individual summands for each $l \in \mathcal{I}$ in (8.64) and split the integrals into

$$I_1^l := \sum_{k=1}^2 \int_{A_l} \alpha_l^* c^l_k(p,t) \nabla_\Gamma Y(p) : \mathrm{Id}\ d\lambda(p,t)$$

and

$$I_2^l := \sum_{k=1}^2 \int_{A_l} \alpha_l^* c^l_k(p,t) \nabla_\Gamma Y(p) : \left(\alpha_l^* \vec{\nu}^l_k \otimes \alpha_l^* \vec{\nu}^l_k\right)\ d\lambda(p,t).$$

Using (8.61) we calculate

$$\begin{aligned}
I_1^l &= \int_{A_l} \left(\sum_{k=1}^2 \alpha_l^* c^l_k(p,t)\right) \nabla_\Gamma Y(p) : \mathrm{Id}\ d\lambda(p,t) \\
&= \int_{A_l} \left(2 - \sum_{k=1}^2 \alpha_l^* \tilde{c}^l_k\right) \nabla_\Gamma Y(p) : \mathrm{Id}\ d\lambda(p,t) \\
&= 2\int_{A_l} \nabla_\Gamma Y(p) : \mathrm{Id}\ d\lambda(p,t) - \int_{A_l} \left(\sum_{k=1}^2 \alpha_l^* \tilde{c}^l_k\right) \nabla_\Gamma Y(p) : \mathrm{Id}\ d\lambda(p,t).
\end{aligned} \tag{8.65}$$

Furthermore, we infer from (8.59) that

$$
\begin{aligned}
I_2^l &= \sum_{k=1}^{2} \int_{\alpha_l(A_l)} c_k^l \nabla_\Gamma Y : \vec{\nu}_k^l \otimes \vec{\nu}_k^l \; d\lambda_l \\
&= \sum_{k=1}^{2} \int_{\alpha_l(A_l)} \left(1 + \tilde{c}_k^l - \sum_{m=1}^{2} \tilde{c}_m^l\right) \left(d_{i,j}^Y\right)_{i,j=1,2} : \vec{\nu}_k^l \otimes \vec{\nu}_k^l \; d\lambda_l \\
&= \int_{\alpha_l(A_l)} \left(d_{i,j}^Y\right)_{i,j=1,2} : \mathrm{Id} \; d\lambda_l + \sum_{k=1}^{2} \int_{\alpha_l(A_l)} \tilde{c}_k^l \left(d_{i,j}^Y\right)_{i,j=1,2} : \vec{\nu}_k^l \otimes \vec{\nu}_k^l \; d\lambda_l \\
&\qquad - \int_{\alpha_l(A_l)} \left(\sum_{m=1}^{2} \tilde{c}_m^l\right) \left(d_{i,j}^Y\right)_{i,j=1,2} : \mathrm{Id} \; d\lambda_l \\
&= \int_{A_l} \nabla_\Gamma Y(p) : \mathrm{Id} \; d\lambda(p,t) - \int_{A_l} \left(\sum_{k=1}^{2} \alpha_l^* \tilde{c}_k^l\right) \nabla_\Gamma Y(p) : \mathrm{Id} \; d\lambda(p,t) \\
&\qquad + \sum_{k=1}^{2} \int_{\alpha_l(A_l)} \left(d_{i,j}^Y\right)_{i,j=1,2} : \left(\tilde{c}_k^l \vec{\nu}_k^l \otimes \vec{\nu}_k^l\right) d\lambda_l. \qquad (8.66)
\end{aligned}
$$

We plug (8.65) and (8.66) into (8.64) and obtain

$$
\begin{aligned}
\langle \delta V_t, Y \rangle &= \sum_{l \in \mathcal{I}} \left(I_1^l + I_2^l\right) \\
&= \sum_{l \in \mathcal{I}} \int_{A_l} \nabla_\Gamma Y(p) : \mathrm{Id} \; d\lambda(p,t) - \sum_{k=1}^{2} \int_{\alpha_l(A_l)} \left(d_{i,j}^Y\right)_{i,j=1,2} : \left(\tilde{c}_k^l \vec{\nu}_k^l \otimes \vec{\nu}_k^l\right) d\lambda_l. \qquad (8.67)
\end{aligned}
$$

Combining (8.63) and (8.67), we have thus proved (8.16).

Chapter 9

Existence of Weak Solutions to the Sharp Interface Model via a Time Discretization Scheme

The construction of weak solutions to geometric evolution equations from time-discrete approximations has been widely used to prove existence results for various equations such as the Stefan problem, the Mullins-Sekerka problem or mean curvature flow. As always with discretization schemes, solutions are obtained as limits of sequences of approximate solutions. The existence of these limits of course depends on the concrete discretization scheme and the considered function spaces. Furthermore it needs to be clarified in which sense the limits could possibly be weak solutions to the geometric evolution equation.

In this section we construct weak solutions to the problem by using a time-discrete scheme that was proposed by Röger in [Rög04] as a generalization of the scheme by Luckhaus and Sturzenhecker in [LS95].

We first introduce the exact setting of the problem in which we work before defining in which sense we speak of weak solutions.

9.1 Problem setting

In contrast to the previous chapter, we work in a slightly different setting. Recall that problem (8.1)–(8.11) was set in an open set $B \subset \mathbb{R}^3$ and its boundary $\partial B =: \Gamma$, where we assumed that Γ was a smooth, closed, two dimensional manifold.

In contrast to this situation we now assume B to be a cuboid in $\mathbb{R}^3$ which is periodic with respect to the first two coordinates, i.e.

$$B := \mathbb{R}^2/2\pi\mathbb{Z}^2 \times [0,1]. \tag{9.1}$$

Consequently, we have

$$\Gamma = \partial B = \mathbb{R}^2/2\pi\mathbb{Z}^2 \times \{0,1\}. \tag{9.2}$$

In this way, functions on the boundary Γ are given as 2π-periodic functions on $\mathbb{R}^2$, while functions in the bulk B only need to be 2π-periodic with respect to the coordinates tangential to Γ.

9.2 Weak solutions

Given a convergent sequence of approximate solutions $\{\phi_h\}_{h>0} \in BV(\Gamma)$, the interface γ in (8.1)–(8.11) is given as the reduced boundary $\partial^*\{\phi = 1\}$ for the limit function ϕ as $h \to 0$. However, this reduced boundary can not necessarily be described by the limit of the surface measures $|\nabla_\Gamma \phi_h|$ since cancellations may occur as discussed in Section 3.2.3. However, using Definition 3.50, we have a suitable concept of a generalized mean curvature vector for the reduced boundary $\partial^*\{\phi = 1\}$. The weak formulation of the sharp-interface limit is based on this definition.

Theorem 9.1. *Assume that B and Γ are given as in* (9.1) *and* (9.2) *respectively. Moreover, let $T > 0$ and suppose that q growth at most linearly, i.e. q fulfils* (2.24). *Then for any*

$$u_0 \in H^1(B), \chi_0 \in BV(\Gamma, \{0,1\}), v_0 \in L^2(\Gamma), \theta_0 \in H^1(\Gamma)$$

such that $\theta_0 = \frac{2}{\delta}(2v_0 - 2\chi_0)$, there exist functions

$$\begin{aligned} &u \in L^2(0,T;H^1(B)), &&\chi \in L^\infty_{w^*}(0,T;BV_{(m_0)}(\Gamma;\{0,1\})), &&v \in L^2((0,T)\times\Gamma),\\ &\theta \in L^2(0,T;H^1(\Gamma)), &&\mu \in L^2(0,T;H^1(\Gamma)). \end{aligned}$$

which are weak solutions to (8.1)–(8.11) *in the following sense: Let $\hat\mu = \mu - \lambda$ where $\lambda = |\Gamma|^{-1}\int_\Gamma \mu$. The equation*

$$-\int_0^T\int_B \nabla u \cdot \nabla \eta = -\int_0^T\int_B u\partial_t\eta + \int_B u_0\eta(0,\cdot) + \int_0^T\int_\Gamma q(u,v)\eta \tag{9.3}$$

holds for all $\eta \in C^\infty([0,T],H^1(B))$ with $\eta(T) = 0$, we have

$$-\int_0^T\int_\Gamma \nabla_\Gamma\theta \cdot \nabla_\Gamma\zeta = -\int_0^T\int_\Gamma v\partial_t\zeta + \int_\Gamma v_0\zeta(0,\cdot) - \int_0^T\int_\Gamma q(u,v)\zeta \tag{9.4}$$

for all $\zeta \in C^\infty([0,T];H^1(\Gamma))$ with $\zeta(T) = 0$ and

$$\int_0^T\int_\Gamma \nabla_\Gamma\hat\mu \cdot \nabla_\Gamma\xi - \int_0^T\int_\Gamma \chi\partial_t\xi = \int_\Gamma \chi_0\xi(0,\cdot) \tag{9.5}$$

for all $\xi \in C^\infty([0,T];H^1(B))$ with $\xi(T) = 0$. Furthermore

$$\theta = \frac{2}{\delta}(2v - 2\chi). \tag{9.6}$$

The essential boundary $\partial^*\{\varphi(\cdot,t) = +1\}$ has for almost all $t \in (0,T)$ a generalized mean curvature vector $H(t) \in L^s(d|\nabla_\Gamma\varphi(t)|)^2, 1 \le s < \infty$ as defined in Definition 3.50 such that

$$H(\cdot,t) = (\hat\mu + \theta + \lambda)\frac{\nabla_\Gamma\chi(\cdot,t)}{|\nabla_\Gamma\chi(\cdot,t)|} \qquad \mathcal{H}^1 - a.e. \text{ for almost all } t \in (0,T). \tag{9.7}$$

9.2.1 Discretization scheme and time discrete solutions

The discretization scheme relies on the BV–formulation of the Gibbs-Thomson law (8.7) given by Luckhaus and Sturzenhecker [LS95], i.e. we construct for each time-step t_h functions $\mu_h \in H^1(\Gamma)$ and χ_h of bounded variation which fulfil

$$\int_\Gamma \operatorname{div}_\Gamma \xi - \frac{\nabla_\Gamma \chi_h}{|\nabla_\Gamma \chi_h|} \cdot D_\Gamma \xi \frac{\nabla_\Gamma \chi_h}{|\nabla_\Gamma \chi_h|} \; d|\nabla_\Gamma \chi_h| = \int_\Gamma \operatorname{div}_\Gamma \left((2\mu_h + \theta_h) \xi \right) \text{ for all } \xi \in C^1(\Gamma, T\Gamma).$$

In the remaining equations (8.1)–(8.6) and (8.8)–(8.11), we substitute the time derivative by its time-discrete approximation $\frac{\chi_h - \chi_{h-1}}{h}$ etc. and formulate the equations in the H^1-sense. The existence of time discrete solutions is granted by the following proposition.

Proposition 9.2. *Let $\tilde{\chi} \in BV_{(m_0)}(\Gamma, \{0,1\}), \tilde{v} \in H^1(\Gamma)$ and $\tilde{u} \in H^1(B)$ be given. Then there exist functions*

$$\chi \in BV_{(m_0)}(\Gamma, \{0,1\}), \quad \hat{\mu} \in H^1_{(0)}(\Gamma), \quad v \in L^2(\Gamma), \quad \theta \in H^1(\Gamma), \quad u \in H^1(B)$$

and a constant $\lambda \in \mathbb{R}$ such that

$$-\int_B \nabla u \cdot \nabla \eta = \int_B \frac{u - \tilde{u}}{h} \eta + \int_\Gamma q(\tilde{u}, \tilde{v}) \eta \text{ for all } \eta \in H^1(B), \tag{9.8}$$

such that

$$\int_\Gamma \left[\nabla_\Gamma \hat{\mu} \cdot \nabla_\Gamma \zeta + \frac{1}{h} (\chi - \tilde{\chi}) \zeta \right] = 0 \text{ for all } \zeta \in H^1(\Gamma), \tag{9.9}$$

and such that θ fulfils

$$\theta = \frac{2}{\delta} (2v - 2\chi) \tag{9.10}$$

together with

$$-\int_\Gamma \nabla_\Gamma \theta \cdot \nabla_\Gamma \zeta = \int_\Gamma \frac{v - \tilde{v}}{h} \zeta - \int_\Gamma q(\tilde{u}, \tilde{v}) \zeta \text{ for all } \zeta \in H^1(\Gamma). \tag{9.11}$$

Moreover,

$$\int_\Gamma \left(\operatorname{div}_\Gamma \xi - \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|} \cdot D\chi \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|} \right) \, d|\nabla_\Gamma \chi| = \int_\Gamma \chi \operatorname{div}_\Gamma ((\hat{\mu} + \lambda + \theta) \xi) \tag{9.12}$$

for all $\xi \in C^\infty(\Gamma, T\Gamma)$.

Furthermore, the estimate

$$\begin{aligned} \int_\Gamma d|\nabla_\Gamma \chi| + \frac{h}{2} \|\nabla_\Gamma \hat{\mu}\|^2_{L^2(\Gamma)} + \frac{\delta}{8} \|\theta\|^2_{L^2(\Gamma)} & \\ + \frac{h}{2} \|\nabla_\Gamma \theta\|^2_{L^2(\Gamma)} + \frac{1}{2} \int_B u^2 + \frac{h}{2} \|\nabla u\|^2_{L^2(B)} & \\ \leq \int_\Gamma d|\nabla_\Gamma \tilde{\chi}| + \frac{\delta}{8} \|\tilde{\theta}\|^2_{L^2(\Gamma)} + Ch \|\tilde{v}\|^2_{L^2(\Gamma)} + Ch \|\tilde{u}\|^2_{H^1(B)} + \frac{1}{2} \int_B \tilde{u}^2 & \end{aligned}$$

holds while the constant λ and $\hat{\mu} \in H^1_{(0)}(\Gamma)$ fulfil additionally

$$|\lambda| \leq C(m_0, \Gamma) \left(1 + \int_\Gamma d|\nabla_\Gamma \chi| \right) \left(\int_\Gamma d|\nabla_\Gamma \chi| + \|\nabla_\Gamma \hat{\mu}\|_{L^2(\Gamma)} \right) \tag{9.13}$$

and

$$\|\hat{\mu} + \lambda\|_{H^1(\Gamma)} \leq c(m_0, \Gamma) \left(1 + \int_\Gamma d|\nabla_\Gamma \chi| \right) \left(\int_\Gamma d|\nabla_\Gamma \chi| + \|\nabla_\Gamma \hat{\mu}\|_{L^2(\Gamma)} \right). \tag{9.14}$$

Let us fix some notation before we start with the proof of Proposition 9.2. We denote by $\delta_\Gamma : L^2(\Gamma) \to H^{-1}(B)$ the operator defined by

$$\langle \delta_\Gamma f, u \rangle_{H^{-1}(B),H^1(B)} := \int_\Gamma f \operatorname{tr} u \, d\mathcal{H}^2$$

for all $f \in L^2(\Gamma)$ and $u \in H^1(B)$.

The proof of Proposition 9.2 now consists of two steps. In the first step we prove for given $\tilde{\chi} \in BV_{(m_0)}(\Gamma, \{0,1\})$, $\tilde{v} \in H^1(\Gamma)$, and $\tilde{u} \in H^1(B)$ the existence of minimizers to a functional

$$\mathcal{F}_h : BV_{(m_0)}(\Gamma, \{0,1\}) \times L^2(\Gamma) \times L^2(B) \to \mathbb{R}$$

defined by

$$\begin{aligned}
\mathcal{F}_h(\chi, v, u) := &\int_\Gamma d\,|\nabla_\Gamma \chi| + \frac{1}{2h}\,\|\chi - \tilde{\chi}\|^2_{H^{-1}(\Gamma)} \\
&+ \frac{1}{2\delta}\int_\Gamma (2v - 2\chi)^2 + \frac{1}{2h}\,\|v - \tilde{v} - hq(\tilde{u}, \tilde{v})\|^2_{H^{-1}(\Gamma)} \\
&+ \frac{1}{2}\int_B u^2 + \frac{1}{2h}\,\|u - \tilde{u} + h\delta_\Gamma q(\tilde{u}, \tilde{v})\|^2_{H^{-1}(B)}\,.
\end{aligned} \tag{9.15}$$

We then prove that these minimizers fulfil the assertion of the proposition.

Lemma 9.3. Let $\tilde{\chi} \in BV_{(m_0)}(\Gamma, \{0,1\}), \tilde{v} \in H^1(\Gamma)$ and $\tilde{u} \in H^1(B)$ be given and let

$$\mathcal{F}_h : BV_{(m_0)}(\Gamma, \{0,1\}) \times L^2(\Gamma) \times L^2(B) \mapsto \mathbb{R}$$

be defined as in (9.15). Then there exist minimizers to $\mathcal{F}_h$.

Proof. The existence of minimizers follows from the direct method of variational calculus. Indeed, $\mathcal{F}_h \geq 0$ is bounded from below. Furthermore, bounded sequences in $BV(\Gamma)$ are precompact in $L^1(\Gamma)$ and the perimeter $\int_\Gamma |\nabla_\Gamma \chi|$ is lower semi-continuous w.r.t. L^1-convergence while bounded sequences in $L^2(\Gamma)$ and $L^2(B)$ at least allow the extraction of weakly converging subsequences in $L^2(\Gamma)$ and $L^2(B)$ respectively. Thus each minimizing sequence $\{(\chi_k, v_k, u_k)\}_{k\in\mathbb{N}}$ has a subsequence such that χ_k converges to a function $\chi \in BV_{(m_0)}(\Gamma, \{0,1\})$ w.r.t the $L^1(\Gamma)$-topology and v_k as well as u_k weakly converge to functions $v \in L^2(\Gamma)$ and $u \in L^2(B)$ respectively w.r.t the corresponding L^2-topology. We remark that the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $L^2(\Gamma)$ because for any $\beta > 0$, Young's inequality and $\int_\Gamma \chi^2 = \int_\Gamma \chi = m_0$ yield

$$\begin{aligned}
\int_\Gamma (v - \chi)^2 = \int_\Gamma (v^2 - 2v\chi + \chi^2) \geq \int_\Gamma v^2 - \beta\int_\Gamma v^2 - \frac{1}{\beta}\int_\Gamma \chi^2 + \int_\Gamma \chi^2 \\
= (1 - \frac{1}{\beta})m_0 + (1 - \beta)\int_\Gamma v^2.
\end{aligned}$$

Given the continuity of the embedding $L^2 \hookrightarrow H^{-1}$, the H^{-1}-norms on Γ and B are weakly lower semi-continuous with respect to the (weak) L^2-topology. Finally, the convex and continuous functional $(\chi, v) \mapsto \int_\Gamma (2v - 2\chi)^2$ posses the same property. Hence $\mathcal{F}_h$ is (weakly) lower semi-continuous and the limit functions (χ, v, u) minimize $\mathcal{F}_h$. $\square$

Proof of Proposition 9.2. We claim that minimizers (χ, u, v) of $\mathcal{F}_h$ as provided by Lemma 9.3 fulfil the assertion of the proposition.

The function $\hat{\mu}$ defined by

$$\hat{\mu} := -(-\Delta_\Gamma)^{-1}\left(\frac{1}{h}(\chi - \tilde{\chi})\right) \tag{9.16}$$

fulfils

$$\int_\Gamma \left[\nabla_\Gamma \hat{\mu} \cdot \nabla_\Gamma \zeta + \frac{1}{h}(\chi - \tilde{\chi})\zeta\right] = 0 \text{ for all } \zeta \in H^1(\Gamma).$$

by the definition of $(-\Delta_\Gamma)^{-1}$.

Furthermore, we define the function θ by

$$\theta := \frac{2}{\delta}(2v - 2\chi).$$

Calculating the first variation of $\mathcal{F}_h$ with respect to v in (χ, u, v) we find

$$\begin{aligned}\frac{\delta \mathcal{F}_h}{\delta v}(\chi, u, v)(\zeta) &= \frac{2}{\delta}\int_\Gamma (2v - 2\chi)\zeta + \frac{1}{h}\int_\Gamma (-\Delta_\Gamma)^{-1}(v - \tilde{v} - hq(\tilde{u}, \tilde{v}))\zeta \\ &= \int_\Gamma \theta\zeta + \frac{1}{h}\int_\Gamma (-\Delta_\Gamma)^{-1}(v - \tilde{v} - hq(\tilde{u}, \tilde{v}))\zeta.\end{aligned}$$

for all $\zeta \in H^1(\Gamma)$. Since (χ, v, u) minimize $\mathcal{F}_h$, the first variation $\frac{\delta \mathcal{F}_h}{\delta v}$ has to vanish in (χ, v, u). Thus one finds

$$\theta = \frac{1}{h}\Delta_\Gamma^{-1}(v - \tilde{v} - hq(\tilde{u}, \tilde{v}))$$

and the function θ solves

$$\Delta_\Gamma \theta = \frac{v - \tilde{v}}{h} - q(\tilde{u}, \tilde{v}) \text{ on } \Gamma \tag{9.17}$$

weakly in $H^1(\Gamma)$, i.e.

$$-\int_\Gamma \nabla_\Gamma \theta \cdot \nabla_\Gamma \zeta = \int_\Gamma \frac{v - \tilde{v}}{h}\zeta - \int_\Gamma q(\tilde{u}, \tilde{v})\zeta \text{ for all } \zeta \in H^1(\Gamma).$$

In particular, this implies $\theta \in H^1(\Gamma)$.

Calculating the first variation $\frac{\delta \mathcal{F}_h}{\delta u}$ of $\mathcal{F}_h$ with respect to u, we find

$$0 = \frac{\delta \mathcal{F}_h}{\delta u}(\chi, u, v)(\eta) = \int_B u\eta + \int_B (-\Delta_N)^{-1}\left(\frac{1}{h}(u - \tilde{u} + h\delta_\Gamma q(\tilde{u}, \tilde{v}))\right)\eta \text{ for all } \eta \in H^1(B)$$

and thus

$$u = \Delta_N^{-1}\left(\frac{1}{h}(u - \tilde{u} + h\delta_\Gamma q(\tilde{u}, \tilde{v}))\right). \tag{9.18}$$

Again we point out that this implies $u \in H^1(B)$. Hence

$$-\int_B \nabla u \cdot \nabla \eta = \int_B \frac{u - \tilde{u}}{h}\eta + \int_\Gamma q(\tilde{u}, \tilde{v})\eta \text{ for all } \eta \in H^1(B)$$

which implies that u is a weak solution to

$$\begin{cases}\Delta u = \frac{1}{h}(u - \tilde{u}) & \text{in } B \\ \nabla u \cdot n = q(\tilde{u}, \tilde{v}) & \text{on } \Gamma.\end{cases}$$

Given the fact that we minimized $\mathcal{F}_h$ over the domain $BV_{(m_0)}(\Gamma,\{0,1\})\times L^2(\Gamma)\times L^2(B)$ we have to consider the condition $\int_\Gamma \chi = m_0$ while calculating the first variation of $\mathcal{F}_h$ with respect to χ. In order to do so, first we calculate this variation with respect to volume preserving variations before determining the Lagrange multiplier associated with $\int_\Gamma \chi = m_0$.

To this end, let $(\Phi_s)_{s\in(-\varepsilon,\varepsilon)}$ be a volume preserving smooth family of diffeomorphisms, $\Phi_s:\Gamma\to\Gamma$, such that $\Phi_0 = \mathrm{Id}$ and $\int_\Gamma \chi\circ\Phi_s^{-1} = m_0$ for all $s\in(-\varepsilon,\varepsilon)$. We denote by $\xi\in C^\infty(\Gamma,T\Gamma)$ the associated vectorfield

$$\xi = \frac{d}{ds}\Big|_{s=0}\Phi_s.$$

Since for all $x\in\Gamma$

$$0 = \frac{d}{ds}\Big|_{s=0}\Phi_s\circ\Phi_s^{-1}(x) = \xi(x) + \frac{d}{ds}\Big|_{s=0}\Phi_s^{-1}(x)$$

we infer

$$\frac{d}{ds}\Big|_{s=0}\Phi_s^{-1} = -\xi. \tag{9.19}$$

Moreover, $(\Phi_s)_{s\in(-\varepsilon,\varepsilon)}$ is volume preserving and together with the foregoing observation and [Mag12, Proposition 17.8] we deduce

$$\frac{d}{ds}\Big|_{s=0}\int_\Gamma \chi\circ\Phi_s^{-1} = \int_\Gamma \chi\,\mathrm{div}_\Gamma\,\xi = 0.$$

As χ and $\chi\circ\Phi_s^{-1}$ are both functions in $BV(\Gamma)$, the perimeter of $\{\chi\circ\Phi_s^{-1}=1\}$ coincides with $\int_\Gamma d\left|\nabla_\Gamma(\chi\circ\Phi_s^{-1})\right|$, see e.g. [AFP00, Theorem 3.36]. Its first variation is

$$\frac{d}{ds}\Big|_{s=0}\int_\Gamma d\left|\nabla_\Gamma(\chi\circ\Phi_s^{-1})\right| = \int_\Gamma\left(\mathrm{div}_\Gamma\,\xi - \frac{\nabla_\Gamma\chi}{|\nabla_\Gamma\chi|}\cdot D\xi\frac{\nabla_\Gamma\chi}{|\nabla_\Gamma\chi|}\right)\,d|\nabla_\Gamma\chi|.$$

If $\partial\{\chi\circ\Phi_s^{-1}=1\}$ is smooth, this follows directly from the divergence theorem for non-tangential vector fields. The result is also true in the $BV-$setting used here, the cumbersome technical details required to carry out the calculations are treated in [Mag12, Theorem 17.5].

Moreover, the characterization of the $H^{-1}-$norm in (3.4) yields

$$\frac{d}{ds}\Big|_{s=0}\frac{1}{2h}\left\|\chi\circ\Phi_s^{-1}-\tilde{\chi}\right\|^2_{H^{-1}(\Gamma)} = \frac{1}{h}\left\langle\frac{\partial}{\partial s}\Big|_{s=0}(\chi\circ\Phi_s^{-1}),(-\Delta_\Gamma)^{-1}(\chi-\tilde{\chi})\right\rangle_{H^{-1},H^1}$$

Using the definition of $\hat{\mu}$ in (9.16) and (9.19) we thus deduce

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\frac{1}{2h}\left\|\chi\circ\Phi_s^{-1}-\tilde{\chi}\right\|^2_{H^{-1}(\Gamma)} &= \langle\nabla_\Gamma\chi\cdot\xi,\hat{\mu}\rangle_{H^{-1},H^1}\\ &= -\int_\Gamma\chi\,\mathrm{div}_\Gamma(\hat{\mu}\xi).\end{aligned}$$

Similarly, [Mag12, Proposition 17.8] and the generalized Gauss-Green formula [AFP00, Theorem 3.36] allows us to calculate

$$\frac{d}{ds}\Big|_{s=0}\frac{1}{2\delta}\int_\Gamma\left(2v-2\left(\chi\circ\Phi_s^{-1}\right)\right)^2 = \int_\Gamma\theta\xi\cdot\frac{\nabla_\Gamma\chi}{|\nabla_\Gamma\chi|}\,d|\nabla_\Gamma\chi| = -\int_\Gamma\chi\,\mathrm{div}_\Gamma\left(\theta\xi\right).$$

We thus deduce

$$0 = \frac{d}{ds}\Big|_{s=0}\mathcal{F}_h(\chi\circ\Phi_s^{-1},u,v) = \int_\Gamma\left(\mathrm{div}_\Gamma\,\xi - \frac{\nabla_\Gamma\chi}{|\nabla_\Gamma\chi|}\cdot D\xi\frac{\nabla_\Gamma\chi}{|\nabla_\Gamma\chi|}\right)\,d|\nabla_\Gamma\chi| - \int_\Gamma\chi\,\mathrm{div}_\Gamma((\hat{\mu}+\theta)\xi). \tag{9.20}$$

Consider now a vectorfield $\hat{\xi} \in C^\infty(\Gamma, \mathbb{R}^2)$ such that

$$\int_\Gamma \chi \operatorname{div}_\Gamma \hat{\xi} \neq 0,$$

in contrast to ξ above. Let $(\hat{\Phi}_s)_{s\in(-\varepsilon,\varepsilon)}$ be a family of smooth diffeomorphisms with $\hat{\Phi}_0 = \mathrm{Id}$ and

$$\frac{d}{ds}\Big|_{s=0} \hat{\Phi}_s = \hat{\xi}.$$

Together with Φ_s as above we can define a function $g : (-\varepsilon, \varepsilon)^2 \to \mathbb{R}$ by

$$g(s,r) := \int \chi \circ (\Phi_s \circ \hat{\Phi}_r)^{-1} - m_0.$$

Since $g(0,0) = 0$ and $\partial_r|_{s=r=0} g(s,r) = \int_\Gamma \chi \operatorname{div} \hat{\xi} \neq 0$ the implicit function theorem yields the existence of an $\varepsilon_1 \leq \varepsilon$ and a function $l : (-\varepsilon_1, \varepsilon_1) \to (-\varepsilon, \varepsilon)$ such that

$$g(s, l(s)) = 0 \text{ for all } s \in (-\varepsilon_1, \varepsilon_1).$$

The family $(\Phi_s \circ \hat{\Phi}_{l(s)})_{s\in(-\varepsilon_1,\varepsilon_1)}$ constitutes therefore a family of volume conserving smooth diffeomorphisms. Hence

$$0 = \frac{d}{ds}\Big|_{s=0} g(s, l(s)) = \frac{d}{ds}\Big|_{s=0} \int \chi \circ (\Phi_s \circ \hat{\Phi}_{l(s)})^{-1} = \int_\Gamma (\operatorname{div}_\Gamma \xi + l'(0) \operatorname{div}_\Gamma \hat{\xi})\chi.$$

We deduce

$$l'(0) = -\left(\int_\Gamma \chi \operatorname{div}_\Gamma \hat{\xi}\right)^{-1} \left(\int_\Gamma \chi \operatorname{div}_\Gamma \xi\right) \tag{9.21}$$

At the same time, (9.20) and the fact that $(\Phi_s \circ \hat{\Phi}_{l(s)})_{s\in(-\varepsilon_1,\varepsilon_1)}$ is a family of volume conserving diffeomorphisms with $\frac{d}{ds}\big|_{s=0} \left(\Phi_s \circ \hat{\Phi}_{l(s)}\right) = \left(\xi + l'(0)\hat{\xi}\right)$ imply

$$\begin{aligned} 0 = \int_\Gamma \left(\operatorname{div}_\Gamma \xi + l'(0) \operatorname{div}_\Gamma \hat{\xi} - \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|} \cdot D\left(\xi + l'(0)\hat{\xi}\right) \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|}\right) d|\nabla_\Gamma \chi| \\ - \int_\Gamma \chi \operatorname{div}_\Gamma \left((\hat{\mu} + \theta)\left(\xi + l'(0)\hat{\xi}\right)\right). \end{aligned}$$

If we plug in (9.21) we obtain

$$\begin{aligned} &\int_\Gamma \left(\operatorname{div} \xi - \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|} \cdot D\chi \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|}\right) d|\nabla_\Gamma \chi| - \int_\Gamma \chi \operatorname{div}_\Gamma((\hat{\mu} + \theta)\xi) \\ &= -l'(0) \left[\int_\Gamma \left(\operatorname{div}_\Gamma \hat{\xi} - \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|} \cdot D\hat{\xi} \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|}\right) d|\nabla_\Gamma \chi| - \int_\Gamma \chi \operatorname{div}_\Gamma((\hat{\mu} + \theta)\hat{\xi})\right] \end{aligned}$$

and with

$$\lambda := \left(\int_\Gamma \chi \operatorname{div}_\Gamma \hat{\xi}\right)^{-1} \left[\int_\Gamma \left(\operatorname{div}_\Gamma \hat{\xi} - \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|} \cdot D\hat{\xi} \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|}\right) d|\nabla_\Gamma \chi| - \int_\Gamma \chi \operatorname{div}_\Gamma((\hat{\mu} + \theta)\hat{\xi})\right]$$

we end up with

$$\int_\Gamma \left(\operatorname{div}_\Gamma \xi - \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|} \cdot D\chi \frac{\nabla_\Gamma \chi}{|\nabla_\Gamma \chi|}\right) d|\nabla_\Gamma \chi| - \int_\Gamma \chi \operatorname{div}_\Gamma((\hat{\mu} + \theta)\xi) = \lambda \int_\Gamma \chi \operatorname{div}_\Gamma \xi. \tag{9.22}$$

Since clearly $\mathcal{F}_h(\chi, v, u) \le \mathcal{F}_h(\tilde{\chi}, \tilde{v}, \tilde{u})$ as (χ, v, u) minimize $\mathcal{F}_h$, (9.17) (which in particular also implies $\theta \in H^1(\Gamma)$) and (9.18) (which in particular also implies $u \in H^1(B)$) allow us to deduce

$$\begin{aligned}
\mathcal{F}_h(\chi, v, u) &= \int_\Gamma d\,|\nabla_\Gamma \chi| + \frac{1}{2h} \|h\nabla_\Gamma \hat{\mu}\|^2_{L^2(\Gamma)} + \frac{\delta}{8} \|\theta\|^2_{L^2(\Gamma)} \\
&\quad + \frac{1}{2h} \|h\nabla_\Gamma \theta\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_B u^2 + \frac{1}{2h} \|h\nabla u\|^2_{L^2(B)} \\
&\le \int_\Gamma d\,|\nabla_\Gamma \tilde{\chi}| + \frac{1}{2\delta} \|2\tilde{v} - 2\tilde{\chi}\|^2_{L^2(\Gamma)} + \frac{1}{2h} \|hq(\tilde{u}, \tilde{v})\|^2_{H^{-1}(\Gamma)} \\
&\quad + \frac{1}{2}\int_B \tilde{u}^2 + \frac{1}{2h} \|h\delta_\Gamma q(\tilde{u}, \tilde{v})\|^2_{H^{-1}(B)} \\
&\le \int_\Gamma d\,|\nabla_\Gamma \tilde{\chi}| + \frac{2}{\delta} \|\tilde{v} - \tilde{\chi}\|^2_{L^2(\Gamma)} + \frac{Ch}{2} \|\tilde{u}\|^2_{L^2(\Gamma)} + \frac{Ch}{2} \|\tilde{v}\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_B \tilde{u}^2
\end{aligned}$$

where we have used that q only growth linearly in $\tilde{u}$ and $\tilde{v}$ and that the embedding $L^2(\Gamma) \hookrightarrow H^{-1}(\Gamma)$ is continuous.

The trace theorem yields $\|\tilde{u}\|_{L^2(\Gamma)} \le C \|\tilde{u}\|_{H^1(B)}$, which implies

$$\begin{aligned}
\mathcal{F}_h(\chi, v, u) &= \int_\Gamma d\,|\nabla_\Gamma \chi| + \frac{h}{2} \|\nabla_\Gamma \hat{\mu}\|^2_{L^2(\Gamma)} + \frac{\delta}{8} \|\theta\|^2_{L^2(\Gamma)} \\
&\quad + \frac{h}{2} \|\nabla_\Gamma \theta\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_B u^2 + \frac{h}{2} \|\nabla u\|^2_{L^2(B)} \\
&\le \int_\Gamma d\,|\nabla_\Gamma \tilde{\chi}| + \frac{\delta}{8} \|\tilde{\theta}\|^2_{L^2(\Gamma)} + C\frac{h}{2} \|\tilde{v}\|^2_{L^2(\Gamma)} + C\frac{h}{2} \|\tilde{u}\|^2_{H^1(B)} + \frac{1}{2}\int_B \tilde{u}^2
\end{aligned}$$

We finish the proof by deducing estimate (9.14). The proof is similar to the proof of Lemma 8.11 and relies on choosing a particular vector field ξ in (9.22). To this end, let $(\rho_\eta)_{\eta>0}$ be a Dirac sequence. We define by

$$\chi_\eta := \chi * \rho_\eta$$

a family of smooth functions with $\left|\chi_\eta - |\Gamma|^{-1} \int_\Gamma \chi_\eta\right| \le 1$ and $|\nabla \chi_\eta| \le \eta^{-1} C(\Gamma)$. We also have

$$\frac{1}{|\Gamma|}\int_\Gamma \chi_\eta \le \frac{m_0}{|\Gamma|}.$$

Let furthermore $\Psi : \Gamma \to \mathbb{R}$ be the solution to

$$\begin{aligned}
\Delta_\Gamma \Psi &= \chi_\eta - |\Gamma|^{-1} \int_\Gamma \chi_\eta \text{ on } \Gamma \\
\int_\Gamma \Psi &= 0
\end{aligned}$$

By Proposition 8.7, the function Ψ satisfies

$$\|\Psi\|_{C^2(\Gamma)} \le C(\Gamma) \left\| \chi_\eta - |\Gamma|^{-1} \int_\Gamma \chi_\eta \right\|_{C^1(\Gamma)} \le \frac{1}{\eta} C(\Gamma).$$

Moreover, assume for a moment that χ is smooth. For y such that $\|y\| \le 1$ we can then always write

$$\chi(p - \eta y) - \chi(p) = -\eta \int_0^1 \nabla_\Gamma \chi(p - s\eta y) \cdot y \; ds.$$

After integrating both sides with respect to p we obtain

$$\int_\Gamma |\chi(p-\eta y)-\chi(p)| \; dp \le \eta \int_0^1 \int_\Gamma |\nabla_\Gamma \chi(p-s\eta y)| \; dp \; ds \le \eta \int_\Gamma |\nabla_\Gamma \chi| \, .$$

Finally, we multiply both sides by $\rho(y)$ and integrate with respect to y to find

$$\int_\Gamma \int_{\mathbb{R}^2} |\chi(p-\eta y)-\chi(p)| \, \rho(y) \; dy \; ds \le \eta \int_\Gamma |\nabla_\Gamma \chi| \, .$$

Since every BV–function can be approximated by a sequence of smooth function (see e.g. [AFP00, Theorem 3.9]), we deduce

$$\|\chi-\chi_\eta\|_{L^1(\Gamma)} \le C(\Gamma)\eta \left(1+\int_\Gamma \; d|\nabla\chi|\right).$$

Choosing $\xi = \nabla_\Gamma \Psi$ yields the estimate

$$\begin{aligned}
\int_\Gamma \chi \operatorname{div}_\Gamma \xi &= \int_\Gamma \chi \left(\chi_\eta - |\Gamma|^{-1} \int_\Gamma \chi_\eta\right) = \left(1-|\Gamma|^{-1}\int_\Gamma \chi_\eta\right) m_0 + \int_\Gamma \chi(\chi_\eta-\chi) \\
&\ge \left(1-\frac{m_0}{|\Gamma|}\right) m_0 - C(\Gamma)\eta\left(1+\int_\Gamma \; d|\nabla_\Gamma\chi|\right) \\
&\ge c(m_0,\Gamma) \qquad (9.23)
\end{aligned}$$

if we choose $\eta = \eta_0(1+\int_\Gamma \; d|\nabla_\Gamma\chi|)^{-1}$ for $\eta_0 = \eta_0(m_0,\Gamma)$ sufficiently small. Plugging these findings in (9.22) implies

$$\begin{aligned}
|\lambda| &= \frac{\left|\int_\Gamma \left(\operatorname{div}_\Gamma \xi - \frac{\nabla_\Gamma\chi}{|\nabla_\Gamma\chi|}\cdot D\chi \frac{\nabla_\Gamma\chi}{|\nabla_\Gamma\chi|}\right) \; d|\nabla_\Gamma\chi| - \int_\Gamma \chi \operatorname{div}_\Gamma((\hat\mu+\theta)\xi)\right|}{\left|\int_\Gamma \chi \operatorname{div}_\Gamma \xi\right|} \\
&\le \frac{\|\Psi\|_{C^2(\Gamma)} \int_\Gamma \; d|\nabla_\Gamma\chi| + 2\,\|\hat\mu\|_{H^1(\Gamma)} \,\|\Psi\|_{C^2(\Gamma)}}{\left|\int_\Gamma \chi \operatorname{div}_\Gamma \xi\right|} \\
&\le \frac{C(\Gamma)\eta^{-1} \int_\Gamma \; d|\nabla_\Gamma\chi| + C(\Gamma)\eta^{-1}\,\|\nabla_\Gamma\hat\mu\|_{L^2(\Gamma)}}{c(m_0,\Gamma)} \\
&\le C(m_0,\Gamma)\left(1+\int_\Gamma \; d|\nabla_\Gamma\chi|\right)\left(\int_\Gamma \; d|\nabla_\Gamma\chi| + \|\nabla_\Gamma\hat\mu\|_{L^2(\Gamma)}\right),
\end{aligned}$$

where we have chosen η exactly as before. In the third step, we used that $\int_\Gamma \hat\mu = 0$ together with Poincaré's inequality and (9.23). Since $\int_\Gamma (\hat\mu+\lambda) = |\Gamma|\,\lambda$ the final estimate (9.14) follows from (9.13) and Poincaré's inequality. □

Proposition 9.2 allows us to construct time discrete solutions

$$(u_h,\chi_h,v_h,\hat\mu_h,\theta_h,\lambda_h) : [0,T] \to H^1(B)\times BV_{(m_0)}(\Gamma,\{0,1\})\times L^2(\Gamma)\times H^1(\Gamma)\times H^1(\Gamma)\times\mathbb{R}.$$

If there are given functions $(u_h^k,\chi_h^k,v_h^k,\hat\mu_h^k,\theta_h^k)$ on some interval $((k-1)h,kh] \subset [0,T]$ where $k\in\mathbb{N}_0$ is such that $(kh,(k+1)h]$ is still a subset of $[0,T]$, we choose $\tilde u(t) = u_h^k(t-h)$, $\tilde\chi(t) = \chi_h^k(t-h)$ and $\tilde v(t) = v_h^k(t-h)$. Proposition 9.2 then yields the existence of functions

$$(u_h^{k+1},\chi_h^{k+1},v_h^{k+1},\hat\mu_h^{k+1},\theta_h^{k+1},\lambda_h^{k+1})$$

on $(kh,(k+1)h]$, such that the equations (9.8) – (9.12) hold.

Starting with $k = 0$ and $(u_h, \chi_h, v_h, \hat{\mu}_h, \theta_h)$ on $(-h, 0]$ given as $\hat{\mu} \equiv 0$ and by the initial data in the case of $u_h, \chi_h, v_h,$ and θ_h, we thus iteratively obtain a time discrete solution $(u_h, \chi_h, v_h, \hat{\mu}_h, \theta_h, \lambda_h)$ on $[0,T]$ by setting

$$(u_h, \chi_h, v_h, \hat{\mu}_h, \theta_h, \lambda_h)(t) := (u_h^k, \chi_h^k, v_h^k, \hat{\mu}_h^k, \theta_h^k, \lambda_h^k) \text{ for } t \in ((k-1)h, kh].$$

Lemma 9.4. Let $(u_h, \chi_h, v_h, \hat{\mu}_h, \theta_h, \lambda_h) : [0,T] \to H^1(B) \times BV_{(m_0)}(\Gamma, \{0,1\}) \times L^2(\Gamma) \times H^1(\Gamma) \times H^1(\Gamma) \times \mathbb{R}$ be the time discrete solutions constructed above. Then the estimates

$$\begin{aligned}\sup_{t\in(0,T)} \int_\Gamma d|\nabla_\Gamma \chi_h(t)| + \|\theta_h\|_{L^\infty(0,T;L^2(\Gamma))} + \|u_h\|_{L^\infty(0,T;L^2(B))} \\ + \|\nabla_\Gamma \hat{\mu}_h\|_{L^2(\Gamma\times(0,T))} + \|\nabla_\Gamma \theta_h\|_{L^2(\Gamma\times(0,T))} + \|\nabla u_h\|_{L^2(B\times(0,T))} \le C(u_0, v_0, \chi_0)\end{aligned} \tag{9.24}$$

and

$$\|\hat{\mu}_h(t) + \lambda_h(t)\|_{H^1(\Gamma)} \le C\left(1 + \|\nabla_\Gamma \hat{\mu}_h(t)\|_{L^2(\Gamma)}\right). \tag{9.25}$$

hold. In particular, the last inequality implies

$$\|\hat{\mu}_h + \lambda_h\|_{L^2(0,T;H^1(\Gamma))} \le C(T).$$

Proof. According to Proposition 9.2 the functions $(u_h, \chi_h, v_h, \mu_h, \theta_h)$ fulfil

$$\begin{aligned}\int_\Gamma d|\nabla_\Gamma \chi_h(t)| + \frac{h}{2}\|\nabla_\Gamma \hat{\mu}_h(t)\|^2_{L^2(\Gamma)} + \frac{\delta}{8}\|\theta_h(t)\|^2_{L^2(\Gamma)} \\ + \frac{h}{2}\|\nabla_\Gamma \theta_h(t)\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_B u_h(t)^2 + \frac{h}{2}\|\nabla u_h(t)\|^2_{L^2(B)} \\ \le \int_\Gamma d|\nabla_\Gamma \chi_h(t-h)| + \frac{\delta}{8}\|\theta_h(t-h)\|^2_{L^2(\Gamma)} + Ch\|v_h(t-h)\|^2_{L^2(\Gamma)} \\ + Ch\|u_h(t-h)\|^2_{H^1(B)} + \frac{1}{2}\int_B u_h(t-h)^2.\end{aligned}$$

Since $\theta_h(t-h) = \frac{2}{\delta}(2v_h(t-h) - 2\chi_h(t-h))$ we deduce

$$\|v_h(t-h)\|^2_{L^2(\Gamma)} \le \left\|\frac{\delta}{4}\theta_h(t-h) + \chi_h(t-h)\right\|^2_{L^2(\Gamma)} \le 2\left\|\frac{\delta}{4}\theta_h(t-h)\right\|^2_{L^2(\Gamma)} + 2\|\chi_h(t-h)\|^2_{L^2(\Gamma)}.$$

The Poincaré inequality for BV–functions in [AFP00, Remark 3.50] and the mass conservation $\int_\Gamma \chi = m_0$ imply $\|\chi_h(t-h)\|^2_{L^2(\Gamma)} \le C\left(1 + \int_\Gamma d|\nabla_\Gamma \chi_h(t-h)|\right)$ and thus

$$\|v_h(t-h)\|^2_{L^2(\Gamma)} \le \frac{\delta^2}{8}\|\theta_h(t-h)\|^2_{L^2(\Gamma)} + C\left(1 + \int_\Gamma d|\nabla_\Gamma \chi_h(t-h)|\right).$$

As a consequence, we obtain

$$\begin{aligned}\int_\Gamma d|\nabla_\Gamma \chi_h(t)| + \frac{h}{2}\|\nabla_\Gamma \hat{\mu}_h(t)\|^2_{L^2(\Gamma)} + \frac{\delta}{8}\|\theta_h(t)\|^2_{L^2(\Gamma)} \\ + \frac{h}{2}\|\nabla_\Gamma \theta_h(t)\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_B u_h(t)^2 + \frac{h}{2}\|\nabla u_h(t)\|^2_{L^2(B)} \\ \le \int_\Gamma d|\nabla_\Gamma \chi_h(t-h)| + \frac{\delta}{8}\|\theta_h(t-h)\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_B u_h(t-h)^2 + Ch\|u_h(t-h)\|^2_{H^1(B)} \\ + Ch\delta^2\|\theta_h(t-h)\|^2_{L^2(\Gamma)} + Ch(1 + \int_\Gamma d|\nabla_\Gamma \chi_h(t-h)|).\end{aligned} \tag{9.26}$$

Summing up (9.26) for $t_k = kh, k = 0, \dots, [t/h]$ and keeping in mind that $(u_h, \chi_h, v_h, \mu_h, \theta_h)$ are constant in t on each subintervall $((k-1)h, kh)$ yields

$$\begin{aligned}
&\int_\Gamma d|\nabla_\Gamma \chi_h(t)| + \frac{\delta}{8}\|\theta_h(t)\|^2_{L^2(\Gamma)} + \|u(t)\|^2_{L^2(B)} \\
&+ \int_0^{h[t/h]} \left(\|\nabla_\Gamma \hat{\mu}_h(t)\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma \theta_h(t)\|^2_{L^2(\Gamma)} + \|\nabla u_h(t)\|^2_{L^2(B)}\right) \\
\le & C(T) + \int_\Gamma d|\nabla_\Gamma \chi_h(0)| + \frac{\delta}{8}\|\theta_h(0)\|^2_{L^2(\Gamma)} + \left\|u^2(0)\right\|_{L^2(B)} \\
&+ C\int_{-h}^{h[t/h]} \left(\int_\Gamma d|\nabla_\Gamma \chi_h(t)| + \|\theta_h(t)\|^2_{L^2(\Gamma)} + \|u_h(t-h)\|^2_{H^1(B)}\right) \\
\le & C(T) + \int_\Gamma d|\nabla_\Gamma \chi_h(0)| + \frac{1}{4}\|\theta_h(0)\|^2_{L^2(\Gamma)} + \left\|u^2(0)\right\|_{L^2(B)} \\
&+ C\int_{-h}^{t} \left(\int_\Gamma d|\nabla_\Gamma \chi_h(s)| + \|\theta_h(s)\|^2_{L^2(\Gamma)} + \|u_h(s)\|^2_{H^1(B)}\right)
\end{aligned}$$

and (9.24) follows from Gronwall's lemma. To deduce the second estimate observe that

$$\|\hat{\mu}_h(t) + \lambda_h(t)\|_{H^1(\Gamma)} \le c(m_0, \Gamma)\left(1 + \int_\Gamma d|\nabla_\Gamma \chi_h(t)|\right)\left(\int_\Gamma d|\nabla_\Gamma \chi_h(t)| + \|\nabla_\Gamma \hat{\mu}_h(t)\|_{L^2(\Gamma)}\right).$$

by Proposition 9.2. Since

$$\sup_{t\in(0,T)} \int_\Gamma |d\nabla_\Gamma \chi_h(t)| \le C$$

by (9.24), this implies (9.25). □

Lemma 9.5. Let $(u_h, \chi_h, v_h, \hat{\mu}_h, \theta_h)$ be as in Lemma 9.4. For any sequence $h \to 0$ there exists a subsequence $\{h_k\}_{k\in\mathbb{N}}$ such that

$$\begin{aligned}
&u_k \rightharpoonup u \text{ in } L^2(0,T;H^1(B)) \text{ and } u_k \to u \text{ in } L^2(0,T;H^s(B)),\ \frac{1}{2} < s < 1, \\
&\chi_k \to \chi \text{ in } L^p((0,T)\times\Gamma) \text{ for all } 1 \le p < \infty, \\
&v_k \rightharpoonup v \text{ in } L^2((0,T)\times\Gamma), \\
&\theta_k \rightharpoonup \theta \text{ in } L^2(0,T;H^1(\Gamma)) \text{ and } \theta_k \to \theta \text{ in } L^p(0,T;L^2(\Gamma)),\ 1 \le p < 2, \\
&\hat{\mu}_k \rightharpoonup \mu \text{ in } L^2(0,T;H^1(\Gamma)), \\
&\lambda_k \rightharpoonup \lambda \text{ in } L^2(0,T).
\end{aligned}$$

where we used the abbreviations $u_k := u_{h_k}, \dots$ etc. The limit functions fulfil

$$\begin{aligned}
&u \in L^2(0,T;H^1(B)), \qquad \chi \in L^\infty_{w^*}(0,T;BV_{(m_0)}(\Gamma;\{0,1\})), \qquad v \in L^2((0,T)\times\Gamma), \\
&\theta \in L^2(0,T;H^1(\Gamma)), \qquad \hat{\mu} \in L^2(0,T;H^1(\Gamma)).
\end{aligned}$$

Proof. First observe that $\{\theta_h\}_{h\in(0,1)}$ is bounded in $L^2(0,T;H^1(\Gamma))$ and that $\theta_h = \frac{2}{\delta}(2v_h - \chi_h)$. We infer from Equation (9.11) and the energy bound (9.24) that for all $k \in \mathbb{N}$ such that $kh < T$

$$\|\tau_{kh} v_h - v_h\|_{L^2(0,T-kh;H^{-1}(\Gamma))} \le khC(T),$$

where $\tau_s f(t) := f(t+s)$ for every $t \in (0, T-s)$. Because of Equation (9.9) we find similarly

$$\|\tau_{kh}\chi_h - \chi_h\|_{L^2(0,T-kh;H^{-1}(\Gamma))} \le khC(T). \tag{9.27}$$

As such, we immediately deduce for every $0 < t_1 < t_2 < T$

$$\sup_{h\in(0,1)} \|\tau_{kh}\theta_h - \theta_h\|_{L^2(t_1,t_2;H^{-1}(\Gamma))} \to 0 \text{ as } h \to 0.$$

Moreover, $\{\theta_h\}_{h\in(0,1)}$ is bounded in $L^2(0,T;H^1(\Gamma))$ by (9.24). Simon's compactness criterion [Sim87, Theorem 6] then implies for a subsequence $\{h_k\}_{k\in\mathbb{N}}$, $h_k \to 0$ as $k \to \infty$, and $1 \le p < 2$ the convergence $\theta_{h_k} \to \theta$ in $L^p(0,T;L^2(\Gamma))$.

We also deduce from the energy estimate in Lemma 9.4 that $\{\chi_h\}_{h\in(0,1)}$ is bounded in $L^\infty(0,T;BV_{(m_0)}(\Gamma;\{0,1\}))$ and together with estimate (9.27) we can again apply Simon's compactness criterion in order to deduce the relative compactness of $\{\chi_h\}_{h\in(0,1)}$ in $L^1((0,T)\times\Gamma)$. Furthermore, $\|\chi_h\|_{L^\infty((0,T)\times\Gamma)} = 1$ and thus $\{\chi_h\}_{h\in(0,1)}$ is relatively compact in $L^p((0,T)\times\Gamma)$ for all $1 \le p < \infty$. Therefore we find a subsequence of $\{h_k\}_{k\in\mathbb{N}}$ (again denoted by h_k) such that $\chi_{h_k} \to \chi$ in $L^p((0,T)\times\Gamma)$.

Moreover, $\{u_h\}_{h\in(0,1)}$ is bounded in $L^2(0,T;H^1(B))$ by Lemma 9.4. Equation (9.28) now implies

$$\|\tau_{kh}u_h - u_h\|_{L^2(0,T-kh;H^{-1}(B))} \le khC(T)$$

and as before we find by [Sim87, Theorem 6] for $1 \le p < 2$ and $1/2 < s < 1$ the strong convergence (up to a subsequence again denoted by h_k) $u_{h_k} \to u$ in $L^p(0,T;H^s(B))$. The weak convergences follow directly from the energy estimates in Proposition 9.2 if one considers again subsequences of $\{h_k\}_{k\in\mathbb{N}}$. □

9.2.2 Equations in the limit and proof of Theorem 9.1

The proof of Theorem 9.1 is split in the following lemmas.

Lemma 9.6. The functions $(u, \chi, v, \theta, \hat{\mu}, \lambda)$ obtained in Lemma 9.5 fulfil (9.3)–(9.6).

Proof. According to (9.8) and the construction of the u_h, the subsequence $\{u_k\}_{k\in\mathbb{N}}$ fulfils

$$-\int_B \nabla u_k(t)\cdot\nabla\eta = \int_B \frac{u_k(t) - u_k(t-h_k)}{h_k}\eta + \int_\Gamma q(u_k(t-h_k), v_k(t-h_k))\eta \tag{9.28}$$

for all $\eta \in H^1(B), t \in [0,T]$. After choosing $\eta \in C^\infty([0,T];H^1(B))$ with $\eta(T) = 0$ and integrating in time, a change of variables in the time variable yields

$$-\int_0^T\int_B \nabla u_k\cdot\nabla\eta = -\int_0^T\int_B u_k\frac{\eta - \eta(\cdot + h_k)}{h_k} - \frac{1}{h}\int_0^h\int_B u_0\eta + \int_0^T\int_\Gamma q(u_k(\cdot - h_k), v_k(\cdot - h_k))\eta \tag{9.29}$$

for all $\eta \in C^\infty([0,T];H^1(B))$ with $\eta(T) = 0$.

From Lemma 9.5 we know $\theta_k \to \theta$ in $L^p(0,T;L^2(\Gamma))$ for all $1 \le p < 2$ and thus up to a subsequence $\theta_k(x) \to \theta(x)$ pointwise almost everywhere in Γ. Because $\chi_k \to \chi$ in $L^p((0,T)\times\Gamma)$ for all $1 \le p < \infty$ we also have $\chi_k(x) \to \chi(x)$ pointwise almost everywhere in Γ for some subsequence. Because $v_k = \frac{\delta}{4}\theta_k + \chi_k$ we find (again for a subsequence),

$$v_k(x) \to v(x) \text{ pointwise almost everywhere in } \Gamma.$$

Moreover, Lemma 9.5 yields for $1 \le p < 2$ and $1/2 < s < 1$ the strong convergence $u_k \to u$ in $L^p(0,T;H^s(B))$. Thus we have $\operatorname{tr} u_k \to \operatorname{tr} u$ in $L^p(0,T;L^2(\Gamma))$ by the continuity of the trace operator. For a suitable subsequence, we directly deduce $u_k(x) \to u(x)$ pointwise almost everywhere on Γ.

Since q growth at most linearly in both arguments, we immediately have

$$\begin{aligned}\|q(u_k,v_k)\|_{L^2((0,T)\times\Gamma)} &\le C(1+\|u_k\|_{L^2((0,T)\times\Gamma)}+\|v_k\|_{L^2((0,T)\times\Gamma)})\\ &\le C(1+\|u_k\|_{L^2(0,T;H^1(B)}+\|v_k\|_{L^2((0,T)\times\Gamma)})\end{aligned}$$

by the trace theorem. Since furthermore

$$\|v_k\|^2_{L^2(\Gamma)} \le \left\|\frac{\delta}{4}\theta_k+\chi_k\right\|^2_{L^2(\Gamma)},$$

Lemma 9.4 and again the Poincaré inequality for BV–functions in [AFP00, Remark 3.50] imply

$$\|v_k\|_{L^2(0,T,L^2(\Gamma))} \le C(T).$$

Thus $q(\operatorname{tr}(u_k),v_k)$ is bounded in the reflexive space $L^2(0,T;L^2(\Gamma))$ and we deduce the existence of a function $\tilde{q} \in L^2(0,T;L^2(\Gamma))$ such that

$$q(\operatorname{tr}(u_k),v_k) \rightharpoonup \tilde{q} \text{ in } L^2(0,T;L^2(\Gamma)).$$

As in the proof of Lemma 4.1, $q(\operatorname{tr}(u_k),v_k)$ converges pointwise almost everywhere to $q(\operatorname{tr}(u),v)$ on $[0,T]\times\Gamma$ thanks to the continuity of q and the convergence results on u_k and v_k above. Since pointwise and weak limit must coincide (if they both exist as in this case), we obtain the weak convergence

$$q(\operatorname{tr}(u_k),v_k) \rightharpoonup q(\operatorname{tr}(u),v) \text{ in } L^2(0,T;L^2(\Gamma)). \tag{9.30}$$

Hence taking the limit in equation (9.29) yields

$$\int_0^T\int_B \nabla u\cdot\nabla\eta = -\int_0^T\int_B u\partial_t\eta - \int_B u_0\eta(0,\cdot) + \int_0^T\int_\Gamma q(u,v)\eta$$

for all $\eta \in C^\infty([0,T],H^1(B))$ with $\eta(T)=0$.

Similarly, (9.11) implies

$$\int_0^T\int_\Gamma \nabla_\Gamma\theta_k\cdot\nabla_\Gamma\zeta = -\int_0^T\int_\Gamma v_k\frac{\zeta-\zeta(\cdot+h_k)}{h_k} - \frac{1}{h}\int_0^h\int_\Gamma v_0\zeta - \int_0^T\int_\Gamma q(u_k(\cdot-h_k),v_k(\cdot-h_k))\zeta$$

for all $\zeta \in C^\infty([0,T];H^1(B))$ with $\eta(T)=0$ and (9.30) allows us to take the limit in this equation to deduce

$$-\int_0^T\int_\Gamma \nabla_\Gamma\theta\cdot\nabla_\Gamma\zeta = -\int_0^T\int_\Gamma v\partial_t\zeta - \int_\Gamma v_0\zeta(0,\cdot) - \int_0^T\int_\Gamma q(u,v)\zeta$$

for all $\zeta \in C^\infty([0,T];H^1(B))$ with $\eta(T)=0$.

Given the convergence results in Lemma 9.5, the limit process in the equations (9.9) and (9.10), leading to (9.5) and (9.6) respectively is straight forward. □

Passing to the limit in (9.12) is more difficult. We expect to find the Gibbs-Thomson law (8.7) in the limit, with the interface γ given as the essential boundary $\partial^*\{\chi = 1\}$. The proof therefore splits in two parts: We have to show that $\partial^*\{\chi = 1\}$ has a generalized curvature as in Definition 3.50 and we have to show that this generalized curvature fulfils the Gibbs-Thomson law.

For the time discrete solutions, let $V_t^k := d|\nabla_\Gamma \chi_k(\cdot,t)|$ be the surface measure of the interfaces $\partial^*\{\chi_k(\cdot,t) = 1\}$, i.e we define

$$V_t^k(\omega) = \int_\Gamma \omega \; d|\nabla_\Gamma \chi_k(\cdot,t)| \;\text{ for } \omega \in C(\Gamma).$$

By (9.12) its first variation δV_t^k is given as

$$\begin{aligned}\delta V_t^k(\xi) &= \int_\Gamma \left(\operatorname{div}_\Gamma \xi - \frac{\nabla_\Gamma \chi_k(\cdot,t)}{|\nabla_\Gamma \chi_k(\cdot,t)|} \cdot D\xi \frac{\nabla_\Gamma \chi_k(\cdot,t)}{|\nabla_\Gamma \chi_k(\cdot,t)|} \right) d|\nabla_\Gamma \chi_k(\cdot,t)| \\ &= \int_\Gamma \chi_k(\cdot,t) \operatorname{div}_\Gamma((\hat{\mu}_k(\cdot,t) + \theta_k(\cdot,t) + \lambda_k(\cdot,t))\xi) \end{aligned} \tag{9.31}$$

for all $\xi \in C^\infty(\Gamma, T\Gamma)$. The first step is to prove that for almost all $t \in (0,T)$ the phase boundary $\partial^*\{\chi = 1\}$ in the limit has a generalized mean curvature which is related to the first variation of V_t^k.

Lemma 9.7. For almost all $t \in (0,T)$ and $1 \le s < \infty$, the phase boundary $\partial^*\{\chi = 1\}$ has a generalized mean curvature $H(t) \in L^s(d|\nabla_\Gamma \chi(\cdot,t)|, T\Gamma)$ which fulfils

$$\int_\Gamma |H(\cdot,t)|^s \, d|\nabla_\Gamma \chi(\cdot,t)| \le C \liminf_{h\to 0} \|\hat{\mu}_h(\cdot,t) + \theta_h(\cdot,t) + \lambda_h(\cdot,t)\|_{H^1(\Gamma)}.$$

Under the assumption that there exists a subsequence $h_k \to 0$ such that

$$\limsup_{k\in\mathbb{N}} \|\hat{\mu}_{h_k}(\cdot,t) + \theta_{h_k}(\cdot,t) + \lambda_{h_k}(\cdot,t)\|_{H^1(\Gamma)} < \infty$$

we obtain furthermore

$$\delta V_t^k(\xi) \xrightarrow{k\to\infty} -\int_\Gamma H(t) \cdot \xi \; d|\nabla_\Gamma \chi(\cdot,t)|$$

for all $\xi \in C^\infty(\Gamma, T\Gamma)$.

Proof. We first observe that Fatou's Lemma and the energy estimate (9.25) imply that $t \mapsto \liminf_{h\to 0} \|\hat{\mu}_h(\cdot,t) + \theta_h(\cdot,t) + \lambda_h(\cdot,t)\|_{H^1(\Gamma)}$ belongs to $L^2(0,T)$. Hence

$$\liminf_{h\to 0} \|\hat{\mu}_h(\cdot,t) + \theta_h(\cdot,t) + \lambda_h(\cdot,t)\|_{H^1(\Gamma)}$$

is finite for almost every $t \in (0,T)$ and we restrict our arguments in the following to those $t \in (0,T)$.

The proof relies now on the convergence result 3.51 for varifolds in $\Omega \subset \mathbb{R}^n$ with mean curvature in $W^{1,p}(\Omega)$ given by Schätzle in [Sch01]. In order to apply this result on $\mathbb{R}^n$ to the varifolds V_t^k on Γ, we localize the necessary calculations in the following way.

As a first step, we introduce suitable varifolds on $\mathbb{R}^2$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/2\pi\mathbb{Z} = \Gamma$ be the quotient map mapping $\mathbb{R}^2$ onto Γ. Moreover, let $\alpha \in \{0,1\}^2$ be a multi-index and define

$$Q_\alpha = [0, 2\pi]^2 - \alpha\pi.$$

To simplify the notation, we set $r_\alpha = \left(\pi|_{Q_\alpha}\right)^{-1}$. For all $\zeta \in C_c(G_1(Q_\alpha))$ we define $U^k_{t,\alpha}$ as

$$U^k_{t,\alpha}(\zeta) = V^k_t\left(r^*_\alpha \zeta\right).$$

Equation (9.31) directly yields

$$\delta U^k_{t,\alpha}(\omega) = \int_{[0,2\pi]^2-\alpha\pi} r^*_\alpha \chi_k(\cdot,t)\,\mathrm{div}((r^*_\alpha \hat{\mu}_k(\cdot,t) + r^*_\alpha \theta_k(\cdot,t) + \lambda_k(\cdot,t))\omega)$$

for all $\omega \in C^1_0(Q_\alpha, \mathbb{R}^2)$, which means that

$$H_{U^k_{t,\alpha}} = r^*_\alpha \hat{\mu}_k(\cdot,t) + r^*_\alpha \theta_k(\cdot,t) + r^*_\alpha \lambda_k(\cdot,t).$$

The map π is isometeric and thus Lemma 9.4 implies that the functions

$$r^*_\alpha \hat{\mu}_k(\cdot,t), \quad r^*_\alpha \theta_k(\cdot,t), \text{ and } r^*_\alpha \lambda_k(\cdot,t)$$

are bounded in $H^1(Q_\alpha)$. As a direct consequence, we deduce that these functions are also bounded in each $W^{1,p}(Q_\alpha)$ for $1 < p \leq 2$ because of $|Q_\alpha| = 4\pi^2$ for all $\alpha \in \{0,1\}^2$. We can thus fix some $p \in (1,2)$ and together with Lemma we obtain (up to a subsequence) the weak convergence

$$r^*_\alpha \hat{\mu}_k(\cdot,t) + r^*_\alpha \theta_k(\cdot,t) + r^*_\alpha \lambda_k(\cdot,t) \rightharpoonup r^*_\alpha \hat{\mu}(\cdot,t) + r^*_\alpha \theta(\cdot,t) + r^*_\alpha \lambda(\cdot,t).$$

By the energy estimate (9.24), the varifolds $U^k_{t,\alpha}$ are all uniformly bounded in k for fixed $\alpha \in \{0,1\}^2$ and $t \in (0,T)$. Starting with $\alpha_0 = (0,0)$ we can hence find a Radon measure U_{t,α_0} and a subsequence $k_j(\alpha_0)$ (depending on α_0) such that $U^{k_j(\alpha_0)}_{t,\alpha_0} \to U_{t,\alpha_0}$ as varifolds on Q_{α_0} as $j \to \infty$.

Continuing with $\alpha_1 = (1,0)$ we can find a subsequence $k_j(\alpha_0,\alpha_1)$ of $k_j(\alpha_0)$ and a Radon measure U_{t,α_1} such that $U^{k_j(\alpha_1)}_{t,\alpha_1} \to U_{t,\alpha_1}$ as varifolds on Q_{α_1} as $j \to \infty$. By the choice of the subsequence $k_j(\alpha_0)$ and the definition of U^k_{t,α_0} and U^k_{t,α_1} respectively, the two measures U_{t,α_0} and U_{t,α_1} must coincide on $Q_{\alpha_0} \cap Q_{\alpha_1}$.

Since there are only finitely many elements $\alpha \in \{0,1\}^2$, we can continue in this way and find a subsequence k_j and a Radon measure for each α such that the requirements of Theorem 3.51 on each Q_α are met and such that the measures $U_{t,\alpha}$ coincide with each other on all overlaps.

Finally, Lemma 9.5 guarantees that

$$r^*_\alpha \chi_{E_{k_j}} \to r^*_\alpha \chi_E \text{ in } L^1(Q_\alpha).$$

As a result, we can apply Theorem 3.51 for each α and deduce that $U_{t,\alpha}$ is an integral varifold fulfilling

$$\partial^*\{r^*_\alpha \chi_E = 1\} \bigcap Q_\alpha \subseteq \operatorname{supp} U_{t,\alpha} \tag{9.32}$$

and

$$\vec{H}_{U_{t,\alpha}} = (r^*_\alpha \hat{\mu}(\cdot,t) + r^*_\alpha \theta(\cdot,t) + r^*_\alpha \lambda(\cdot,t))\,\nu_E \quad m_{U_{t,\alpha}} - \text{almost everywhere on } \operatorname{supp} U_{t,\alpha}$$

Moreover, $\vec{H}_{U_{t,\alpha}}$ is the generalized mean curvature vector of $\partial^*\{r^*_\alpha \chi_E = 1\} \cap Q_\alpha$.

Choose $z_0, z_i \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ in such a way that

$$\operatorname{supp} z_0|_{[0,2\pi]} \subset (0,2\pi), \qquad \operatorname{supp} z_1|_{[-\pi,\pi]} \subset (-\pi,\pi)$$

and

$$z_0(p_1) + z_1(p_1) = 1 \text{ for all } p_1 \in \mathbb{R}/2\pi\mathbb{Z}.$$

With $\alpha \in \{0,1\}^2$ as above, we introduce cut-off functions z_α on Γ by setting

$$z_\alpha(p) = \prod_{j=1}^{2} z_{\alpha_j}(p_j) \text{ for } p = (p_1, p_2) \in \Gamma.$$

Note that $\sum_{\alpha\in\{0,1\}^2} z_\alpha(p) = 1$ for all $p \in \Gamma$.

We define the varifold V_t on Γ by

$$V_t(\psi) = \sum_{\alpha\in\{0,1\}^2} U_{t,\alpha}(\pi^{-1,*}(z_\alpha\psi)) \text{ for all } \psi \in C_0(G_1(\Gamma)).$$

One readily calculates

$$V_t^{k_j} \to V_t \text{ as varifolds}$$

and from (9.32) we deduce

$$\partial^*\{\chi = 1\} \subseteq \operatorname{supp} V_t.$$

Given that $\delta V_t^{k_j}$ is linear in ξ, we find

$$\begin{aligned}
\delta V_t^{k_j}(\xi) &= \delta V_t^{k_j}\left(\sum_{\alpha\in\{0,1\}^2} z_\alpha \xi\right) \\
&= \sum_{\alpha\in\{0,1\}^2} \int_\Gamma \left(\operatorname{div}_\Gamma(z_\alpha\xi) - \frac{\nabla_\Gamma \chi_{k_j}(\cdot,t)}{|\nabla_\Gamma \chi_{k_j}(\cdot,t)|} \cdot D(z_\alpha\xi) \frac{\nabla_\Gamma \chi_{k_j}(\cdot,t)}{|\nabla_\Gamma \chi_{k_j}(\cdot,t)|}\right) d\left|\nabla_\Gamma \chi_{k_j}(\cdot,t)\right|.
\end{aligned}$$

Because the functions z_α have compact support, we use the definition of the varifolds $U_{t,\alpha}^k$ to see that we can transform each summand to obtain

$$\begin{aligned}
&\lim_{j\to\infty} \delta V_t^{k_j}(\xi) \\
&= \lim_{j\to\infty} \sum_{\alpha\in\{0,1\}^2} \int_{\Gamma \cap \operatorname{supp} z_\alpha} \left(\operatorname{div}_\Gamma(z_\alpha\xi) - \frac{\nabla_\Gamma \chi_{k_j}(\cdot,t)}{|\nabla_\Gamma \chi_{k_j}(\cdot,t)|} \cdot D(z_\alpha\xi) \frac{\nabla_\Gamma \chi_{k_j}(\cdot,t)}{|\nabla_\Gamma \chi_{k_j}(\cdot,t)|}\right) d\left|\nabla_\Gamma \chi_{k_j}(\cdot,t)\right| \\
&= \lim_{j\to\infty} \sum_{\alpha\in\{0,1\}^2} \delta U_{t,\alpha}^{k_j}(\pi^{-1,*}(z_\alpha\xi)) \\
&= \sum_{\alpha\in\{0,1\}^2} \int_{[0,2\pi]^2-\alpha\pi} r_\alpha^*\chi(\cdot,t) \operatorname{div}((r_\alpha^*\hat\mu(\cdot,t) + r_\alpha^*\theta(\cdot,t) + \lambda(\cdot,t))\pi^{-1,*}(z_\alpha\xi)) \\
&= \int_\Gamma \chi(\cdot,t) \operatorname{div}_\Gamma((\hat\mu(\cdot,t) + \theta(\cdot,t) + \lambda(\cdot,t))\xi).
\end{aligned}$$

This proves that the varifold V_t has a mean curvature vector $\vec H_{V_t}$ such that

$$\vec H_{V_t} = (\hat\mu(\cdot,t) + \theta(\cdot,t) + \lambda(\cdot,t))\,\nu_E \quad m_{V_t} - \text{almost everywhere on } \operatorname{supp} V_t$$

where $\nu_E = \frac{\nabla_\Gamma \chi_E}{|\nabla_\Gamma \chi_E|}$ denotes the generalized normal of $\partial^* E$, which is set to be equal to 0 outside of $\partial^* E$. The crucial observation in Proposition 3.49 proved by Röger in [Rög04] is that the mean curvature vector $\vec{H}_{V_t}$ on $\partial^*\{\chi = 1\}$ only depends on $\chi(\cdot,t)$, not on the varifold V_t. As such it is in particular independent from the choice of the subsequence $\{k_j\}_{j\in\mathbb{N}}$ above, since we have

$$\partial^*\{\chi_E = 1\} \subseteq \operatorname{supp} V_t,$$

regardless of the chosen subsequence.

Therefore, we conclude that $H(t) := \vec{H}_{V_t} \cdot \frac{\nabla_\Gamma \chi_E}{|\nabla_\Gamma \chi_E|}$ is the generalized mean curvature vector of $\partial^*\{\chi = 1\}$, which completes the proof. $\square$

Proof of Theorem 9.1. Following Lemma 9.6 and 9.7, it remains to show $H(\cdot,t) = \hat{\mu} + \theta + \lambda$ in $L^2(0,T,H^1(\Gamma))$.

By Lemma 9.7, the operator $T(t) : C_0^1(\Gamma, T\Gamma) \to \mathbb{R}$ defined by

$$\langle T(t), \xi\rangle := \int_\Gamma -H(x,t)\cdot\xi \; d|\nabla_\Gamma \chi(x,t)|$$

exists for almost every $t \in (0,T)$. Let furthermore $T^h(t)$ be given by

$$\langle T^h(t), \xi\rangle := \delta V_t^h(\xi).$$

Proposition 9.2 now yields

$$\langle T^h(t), \xi\rangle = \int_\Gamma \chi_h \operatorname{div}_\Gamma\left((\hat{\mu}_h + \theta_h + \lambda_h)\,\xi\right)$$

and by Lemma 9.5 there exist subsequences $\{h_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\int_0^T \langle T^{h_k}(t), \xi(\cdot,t)\rangle\; dt = \int_0^T\int_\Gamma \chi \operatorname{div}_\Gamma\left((\hat{\mu} + \theta + \lambda)\,\xi\right)$$

Next we show that

$$\lim_{k\to\infty}\int_0^T \langle T^{h_k}(t), \xi(\cdot,t)\rangle\; dt = \int_0^T \langle T(t), \xi(\cdot,t)\rangle\; dt,$$

using Lemma 9.7 and following the arguments in [AR09, Lemma 4.6]. Lemma 9.7 yields the desired convergence pointwise almost everywhere in time under a boundedness assumption on $\hat{\mu}_{h_k}(t) + \theta_{h_k}(t) + \lambda_{h_k}(t)$. To apply this lemma, we introduce $T_\alpha^{h_k}(t) : (0,T) \to C_0^1(\Gamma, T\Gamma)'$ defined by

$$\langle T_\alpha^{h_k}(t), \xi(\cdot,t)\rangle = \begin{cases} \langle T^{h_k}(t), \xi(\cdot,t)\rangle, & \text{if } \|\hat{\mu}_{h_k}(t) + \theta_{h_k}(t) + \lambda_{h_k}(t)\|_{H^1(\Gamma)} \le \alpha, \\ \langle T(t), \xi(\cdot,t)\rangle & \text{else.} \end{cases}$$

Now Lemma 9.7 allows us to fix any $\xi \in L^2(0,T;C_0^1(\Gamma,T\Gamma))$ and to obtain

$$\lim_{k\to\infty}\langle T_\alpha^{h_k}(t), \xi(\cdot,t)\rangle = \langle T(t), \xi(\cdot,t)\rangle \tag{9.33}$$

for almost all $t \in (0,T)$ since we have that either $\langle T_\alpha^{h_k}(t), \xi(\cdot,t)\rangle = \langle T(t), \xi(\cdot,t)\rangle$ by definition or the boundedness condition in the lemma is fulfiled. Because of

$$\left|\langle T^{h_k}(t), \xi(\cdot,t)\rangle\right| \le C\,\|\hat{\mu}_{h_k}(t) + \theta_{h_k}(t) + \lambda_{h_k}(t)\|_{H^1(\Gamma)}\,\|\xi(\cdot,t)\|_{C_0^1(\Gamma,T\Gamma)}$$

we can deduce

$$\begin{aligned}\left|\langle T_\alpha^{h_k}(t), \xi(\cdot,t)\rangle\right| &\le C\,\|\xi(t)\|_{C_0^1(\Gamma)}\left(\alpha + \|T(t)\|_{C^0(\Gamma,T\Gamma)'}\right)\\ &\le C\,\|\xi(t)\|_{C_0^1(\Gamma)}\left(\alpha + \|H(t)\|_{L^s(d|\nabla\xi(x,t)|)}\right),\end{aligned} \tag{9.34}$$

which in particular gives an $L^1(0,T)$–majorant due to Lemma 9.7. Because of the pointwise convergence for almost all $t \in (0,T)$ in (9.33), we invoke Lebesgues Dominated Convergence Theorem to obtain for all $\alpha > 0$

$$\lim_{k\to\infty}\int_0^T \langle T_\alpha^{h_k}(t), \xi(\cdot,t)\rangle\ dt = \int_0^T \langle T(t), \xi(\cdot,t)\rangle\ dt. \tag{9.35}$$

In order to complete the proof, we study all sets

$$Z_\alpha^h := \left\{t \in (0,T)\,\middle|\,\|\hat\mu_{h_k}(t) + \theta_{h_k}(t) + \lambda_{h_k}(t)\|_{H^1(\Gamma)} > \alpha\right\},$$

i.e. all the set of all times $t \in (0,T)$ for which we set $\langle T_\alpha^{h_k}(t), \xi(\cdot,t) = \langle T(t), \xi(\cdot,t)\rangle$. Recall that $\|\hat\mu_{h_k} + \theta_{h_k} + \lambda_{h_k}\|_{L^2(0,T;H^1(\Gamma))}$ is bounded uniformly in $h_k > 0$ by Lemma 9.4. Thus

$$\begin{aligned}\left|Z_\alpha^{h_k}\right| = \int_{Z_\alpha^{h_k}} 1\ dt &\le \int_{Z_\alpha^{h_k}} \frac{1}{\alpha^2}\,\|\hat\mu_{h_k}(t) + \theta_{h_k}(t) + \lambda_{h_k}(t)\|^2_{H^1(\Gamma))}\ dt\\ &\le \frac{1}{\alpha^2}\,\|\hat\mu_{h_k} + \theta_{h_k} + \lambda_{h_k}\|_{L^2(0,T;H^1(\Gamma))} \le \frac{C}{\alpha^2}.\end{aligned}$$

Next observe that $T_\alpha^{h_k}(t) = T^{h_k}(t)$ for all $t \in (0,T)\setminus Z_\alpha^{h_k}$ and calculate

$$\begin{aligned}&\left|\int_0^T \left\langle T^{h_k}(t) - T_\alpha^{h_k}(t), \xi(\cdot,t)\right\rangle\ dt\right| \le \int_{Z_\alpha^{h_k}} \left|\left\langle T^{h_k}(t) - T(t), \xi(\cdot,t)\right\rangle\right|\ dt\\ &\le \left(\int_{Z_\alpha^{h_k}} \|\xi(\cdot,t)\|_{C_0^1(\Gamma,T\Gamma)}\right)^{1/2}\left(\left\|T^{h_k}\right\|_{L^2(0,T;C_0^1(\Gamma,T\Gamma))'} + \|T\|_{L^2(0,T;C_0^1(\Gamma,T\Gamma))'}\right)\end{aligned}$$

which yields

$$\lim_{\alpha\to\infty}\sup_{k\in\mathbb{N}}\int_0^T \left\langle T^{h_k}(t) - T_\alpha^{h_k}(t), \xi(\cdot,t)\right\rangle\ dt = 0 \tag{9.36}$$

since $\left\|T^{h_k}\right\|_{L^2(0,T;C_0^1(\Gamma,T\Gamma))'}$ and $\|T\|_{L^2(0,T;C_0^1(\Gamma,T\Gamma))'}$ are bounded uniformly in h_k by Lemma 9.7 and the calculations that led to (9.34). Hence (9.35) implies

$$\begin{aligned}&\int_0^T\int_\Gamma \chi\,\mathrm{div}_\Gamma\left((\hat\mu + \theta + \lambda)\,\xi\right)\ dx\ dt = \lim_{k\to\infty}\int_0^T \langle T^{h_k}(t), \xi(\cdot,t)\rangle\ dt\\ &= \lim_{k\to\infty}\left[\int_0^T \langle T_\alpha^{h_k}(t), \xi(\cdot,t)\rangle + \langle T^{h_k}(t) - T_\alpha^{h_k}(t), \xi(\cdot,t)\rangle\ dt\right]\\ &= \int_0^T \langle T(t), \xi(\cdot,t)\rangle\ dt + \lim_{k\to\infty}\left[\int_0^T \langle T^{h_k}(t) - T_\alpha^{h_k}(t), \xi(\cdot,t)\rangle\ dt\right].\end{aligned}$$

Due to (9.36), the remaining term on the right hand-side vanishes for $\alpha \to \infty$ since the convergence in (9.36) is uniform in $k \in \mathbb{N}$. Therefore we deduce

$$\begin{aligned}\int_0^T\int_\Gamma \chi\,\mathrm{div}_\Gamma\left((\hat\mu + \theta + \lambda)\,\xi\right)\ dx\ dt &= \lim_{k\to\infty}\int_0^T \langle T^{h_k}(t), \xi(\cdot,t)\rangle\ dt\\ &= \int_0^T \langle T(t), \xi(\cdot,t)\rangle\ dt.\end{aligned}$$

Furthermore,

$$\int_\Gamma \chi(\cdot,t)\,\mathrm{div}_\Gamma\left((\hat{\mu}(\cdot,t)+\theta(\cdot,t)+\lambda(\cdot,t))\,\zeta(\cdot,t)\right)\,dx = \langle T(t),\zeta\rangle$$

holds for almost all $t \in (0,T)$ and all $\zeta \in C_0^1(\Gamma, T\Gamma)$. Finally, we set $\nu(\cdot,t) = \frac{\nabla_\Gamma \chi(\cdot,t)}{|\nabla_\Gamma \chi(\cdot,t)|}$ on $\partial^*\{\chi(\cdot,t)=1\}$ and the divergence theorem yields

$$\int_\Gamma (\hat{\mu}(\cdot,t)+\theta(\cdot,t)+\lambda(\cdot,t))\nu(\cdot,t)\cdot\zeta\, d|\nabla_\Gamma\chi(\cdot,t)| = \int_\Gamma H(\cdot,t)\cdot\xi\, d|\nabla_\Gamma\chi(\cdot,t)|,$$

which concludes the proof. ☐

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